On the Mathematical Theory of Vibration

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In the paper systems of differential equations of the form $\dot{x} = f(x, u_n, t)$ whose right-hand side involves vibration are treated. A new approach to obtaining the limit system as $u_n \to 0$ is suggested. This approach cannot be reduced to the averaging methods presented in the papers [1-3] and yields limit systems both for the case of right-hand sides that are continuous with respect to the phase variable and for discontinuous right-hand sides (for instance, for problems with dry friction). Here the result is written not by means of a passage to the limit but in explicit form and describes the entire class of limit systems. An application of the presented theory to model vibration problems in mechanics is indicated.

PART I. CONTINUOUS CASE

INTRODUCTION

We consider systems of the form

$$\dot{x} = f(x, u_n, t), \quad x \in \mathbb{R}^{d(x)}, \quad u \in \mathbb{R}^m, \quad t \in [t_0, t_1],$$

where $f$ is a continuous function with respect to $x, u$ that satisfies the Lipschitz condition with respect to the variable $x$ and $d(x)$ is the dimension of the vector $x$.

The problem is to write out the limit system. Formally, this problem can be reduced to the one considered in the papers [1-3], where a small parameter enters the right-hand side of the system in an arbitrary way (in (1), the role of the small parameter is played by the
sequence $u_n$). Since the problem in the above papers is stated in general form, the result is not written out in explicit form but by means of a passage to the limit. It turns out that in our special case, in which the right-hand side has the form (1) and $u_n \overset{L_{\infty}-\text{weakly}}{\rightarrow} 0$, we can write out the limit system in explicit form by regarding the weak convergence of the argument as the weak convergence of measures in another conjugate space. This makes it possible to characterize the entire class of limit systems in a form not depending on the character of the approximation $u_n \overset{L_{\infty}-\text{weakly}}{\rightarrow} 0$ and on the specific expression for the functions $u_n(t)$. This class of limit systems can be written in the form

$$
\dot{x}(t) = \int_{\mathbb{R}^m} f(x, u, t) \mu(du, t)
$$

(2)

under the following assumptions:

$$
\int u \mu(du, t) = 0, \quad \int \mu(du, t) = 1.
$$

(3)

where, for each $t$, $\mu(du, t)$ is a Radon measure whose support belongs to some compact set. Thus, for the parameter of the limit system, we have a mixed strategy. This approach turns out to be very useful, e.g., in vibration problems in mechanics, since we can choose the parameter $\mu(du, t)$ so that the limit system has the desired properties, and then reconstruct the vibration from $\mu$. Moreover, in contrast to the methods of [1–3], this description of the limit system can be generalized to the case in which the right-hand side of system (1) is discontinuous with respect to the phase variable. This will be presented in the second part of the paper.

§1. MIXED STRATEGIES

We consider the class of mixed strategies $\mu(du, t), t \in [t_0, t_1], u \in \mathbb{R}^m$, where $\mu(du, t)$ are unit positive Radon measures, i.e., probability measures, that depend on the parameter $t$ and the following conditions are satisfied:

1) $\mu(du, t)$ have common compact support $K_u$, i.e., $\mu(K_u, t) = 1$ for any $t \in [t_0, t_1]$;
2) the function $t \mapsto \int_{\mathbb{R}^m} f(u) \mu(du, t), t \in [t_0, t_1]$, is measurable for any continuous function $f(u)$ of the variable $u$.

Denote this class of mixed strategies by $V$. For a chosen compact set $K_u$, denote by $V_{K_u}$ the set of measures belonging to $V$ for which $K_u$ is a common compact support, i.e., the set of $\mu$ such that $\mu(K_u) = 1$.

We say that $\mu_n \overset{\text{weakly}}{\rightarrow} \mu$ if the following conditions hold:

a) there exists a compact set $K_u$ such that $\mu_n(du, t), \mu(du, t) \in V_{K_u}$;

b) $\int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} f(u, t) \mu_n(du, t) \rightarrow \int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} f(u, t) \mu(du, t) \quad \forall f(u, t) \in L_u$, where $L_u$ is the space of functions $f(u, t)$ such that

i) $f(\cdot, t)$ is continuous for every $t \in [t_0, t_1]$,
ii) $f(u, \cdot)$ is measurable for every $u \in K_u$, and
iii) for each of the sets $K_u$ we have $\max_{u \in K_u} |f(u, t)| \in L_1[t_0, t_1]$.

It follows from conditions i) and ii) that the function $\max_{u \in K_u} |f(u, t)|$ is measurable. The function $\psi(t) = \int_{\mathbb{R}^m} f(u, t) \mu(du, t)$ is measurable as well, which follows from condition 2) and conditions i)–iii). As is known, under these assumptions the set $V_{K_u}$ is a metrizable compact set in the weak topology.
Consider the set \( V_{K_u}^0 \subset V_{K_u} \) defined by \( V_{K_u}^0 = \{ E_{u_\star}(t) : E_{u_\star}(t)(du, t) = \delta(u - u_\star(t)) du \} \), where \( u_\star(t) \) is a bounded measurable function and \( \delta(x) \) is a delta function. We can readily prove that the set \( V_{K_u}^0 \) is weakly dense in the set \( V_{K_u} \), i.e., \( \bar{V}_{K_u}^0 = V_{K_u} \). The following assertion holds.

**Theorem 1.** If \( u_n \xrightarrow{L_\infty} 0 \) and \( E_{u_n}(du, t) \xrightarrow{\text{weakly}} \mu(du, t) \), then \( \int_{\mathbb{R}^m} u\mu(du, t) = 0 \).

Conversely, if \( \int_{\mathbb{R}^m} u\mu(du, t) = 0 \) and if a sequence \( u_n \) is such that \( E_{u_n}(du, t) \xrightarrow{\text{weakly}} \mu(du, t) \), then we have \( u_n \xrightarrow{L_\infty} 0 \).

**Proof.** 1) Assume that \( u_n \xrightarrow{L_\infty} 0 \), i.e., \( \int_{t_0}^{t_1} h(t)u_n(t) dt \to 0 \) for any \( h(t) \in L_\infty \). Then
\[
\int_{t_0}^{t_1} h(t)u_n(t) dt = \int_{t_0}^{t_1} h(t) dt \int_{\mathbb{R}^m} u E_{u_n}(du, t) \to \int_{t_0}^{t_1} h(t) dt \int_{\mathbb{R}^m} u \mu(du, t) = 0
\]
for any \( h(t) \in L_\infty \). This implies the relation \( \int_{\mathbb{R}^m} u\mu(du, t) = 0 \).

2) Assume that \( \int_{\mathbb{R}^m} u\mu(du, t) = 0 \). Then \( \int_{t_0}^{t_1} h(t) dt \int_{\mathbb{R}^m} u\mu(du, t) = 0 \) for any \( h(t) \in L_\infty \). Since the measures \( E_{u_n}(du, t) \) form a weakly dense subset of the space \( V_{K_u} \), it follows that for each of the measures \( \mu(du, t) \) there exists a sequence \( E_{u_n} \to \mu(du, t) \), i.e.,
\[
\lim_{n \to \infty} \int_{t_0}^{t_1} h(t) u_n(t) dt = \lim_{n \to \infty} \int_{t_0}^{t_1} h(t) dt \int_{\mathbb{R}^m} u E_{u_n}(du, t) = 0 \Rightarrow \int_{t_0}^{t_1} h(t) u(t) dt = 0 \Rightarrow u_n \xrightarrow{L_\infty} 0.
\]

This completes the proof of Theorem 1.

**Theorem 2.** Let \( u_n(t), \widehat{u}_n(t) \in K_u \) for any \( t \in [t_0, t_1] \) and let the following condition hold:
\[
\int_{t_0}^{t_1} |u_n(t) - \widehat{u}_n(t)| dt \to 0.
\]

If \( E_{u_n}(du, t) \xrightarrow{\text{weakly}} \mu(du, t) \), then \( E_{\widehat{u}_n}(du, t) \xrightarrow{\text{weakly}} \mu(du, t) \).

**Proof.** It follows from the definition of the weak convergence of measures (condition b)) that
\[
\int_{t_0}^{t_1} f(u_n, t) dt \to \int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} f(u, t) \mu(du, t) \quad \forall f(u, t) \in L_u.
\]

We must prove that
\[
\int_{t_0}^{t_1} f(\widehat{u}_n, t) dt \to \int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} f(u, t) \mu(du, t).
\]

To this end, it suffices to show that
\[
\int_{t_0}^{t_1} |f(u_n, t) - f(\widehat{u}_n, t)| dt \to 0.
\]

According to condition (4), the sequence \( \widehat{u}_n(t) \) converges to the sequence \( u_n(t) \) with respect to the norm of the space \( L_1 \), and hence is convergent in measure. As is known, from any sequence convergent in measure we can extract a subsequence convergent almost everywhere. Therefore, from any subsequence convergent in measure we can extract a subsequence convergent almost everywhere. These facts, together with the continuity of the function \( f(u, t) (f(u, t) \in L_u) \) with respect to \( u \), imply relation (5). This completes the proof of Theorem 2.
§2. STATEMENT OF THE MAIN RESULTS

Let $K_x$ and $K_u$ be compact sets in $\mathbb{R}^d(x)$ and $\mathbb{R}^m$, respectively. Denote by $y$ a pair $(x, u) = y$, where $x \in K_x$, $u \in K_u$, and $y \in K_x \times K_u$.

Consider the space of functions $f(x, u, t)$ such that

1) $f(x, u, t) = f(y, t) \in \mathcal{L}_y$, where $\mathcal{L}_y$ is defined similarly to $\mathcal{L}_u$ (see §1, i–iii); in what follows we denote the space $\mathcal{L}_y$ by $\mathcal{L}_{xy}$;

2) there exists a measurable full-measure set $\mathcal{E} [t_0, t_1]$ such that, for each $t \in \mathcal{E}$, the function $f(x, u, t)$ satisfies the Lipschitz condition with respect to the variable $x$ with constant $\mathcal{L}(t)$, i.e., $|f(x_1, u, t) - f(x_2, u, t)| \leq \mathcal{L}(t)|x_1 - x_2|$ for all $x_1, x_2 \in K_x$, $u \in K_u$, and $t \in \mathcal{E}$, where $\mathcal{L}(t) \in L_1$.

Assume that $\mu_n \xrightarrow{\text{weakly}} \mu$. Consider the following systems:

$S_n : \dot{x}(t) = \int_{\mathbb{R}^m} f(x, u, t)\mu_n(du, t), \quad (6)$

$S : \dot{x}(t) = \int_{\mathbb{R}^m} f(x, u, t)\mu(du, t). \quad (7)$

We say that $S_n \rightarrow S$ provided that the following assertions hold.

**Assertion 1.** Let $f(x, u, t) \in \mathcal{L}_{xy}$, let $\mu_n \xrightarrow{\text{weakly}} \mu$, and let there exist a sequence of solutions $x_n(t)$ of system $S_n$ that converges to $x^0(t)$ with respect to the norm of the space $C[t_0, t_1]$. Then $x^0(t)$ is a solution of the system $S$:

$$\dot{x}^0(t) = \int_{\mathbb{R}^m} f(x^0, u, t)\mu(du, t).$$

**Assertion 2.** Let $f(x, u, t) \in \mathcal{L}_{xy}$, let $x^0(t)$ be a solution of the system $S$, let $\mu_n \xrightarrow{\text{weakly}} \mu$, and let $x_n \rightarrow x^0(t_0)$. Then, beginning with some $n$, there exists a solution $x_n(t)$ of the system $S_n (x_n(t_0) = \xi_n)$ such that $x_n(t) \rightarrow x^0(t)$ with respect to the norm of the space $C[t_0, t_1]$.

§3. PROOF OF THE RESULTS

To prove the above statements, we need the following theorems.

**Theorem 3.** Let $f(u, t) \in \mathcal{L}_u$ and $\mu_n(du, t) \xrightarrow{\text{weakly}} \mu(du, t)$. Then

$$\int_{\mathbb{R}^m} f(u, t)\mu_n(du, t) \xrightarrow{L_\infty\text{-weakly}} \int_{\mathbb{R}^m} f(u, t)\mu(du, t),$$

that is,

$$\int_{t_0}^{t_1} h(t)\psi_n(t) dt \xrightarrow{} \int_{t_0}^{t_1} h(t)\psi(t) dt \quad \forall h \in L_\infty,$$

where $\psi_n(t) = \int_{\mathbb{R}^m} f(u, t)\mu_n(du, t)$ and $\psi(t) = \int_{\mathbb{R}^m} f(u, t)\mu(du, t)$.

**Proof.** Since $h(t)$ does not depend on $u$, it follows that

$$\int_{t_0}^{t_1} h(t) \int_{\mathbb{R}^m} f(u, t)\mu_n(du, t) dt = \int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} h(t)f(u, t)\mu_n(du, t)$$

$$= \int_{t_0}^{t_1} dt \int_{\mathbb{R}^m} f_1(u, t)\mu_n(du, t),$$
where \( f_1(u,t) = h(t)f(u,t) \). It follows from the relation \( h(t) \in L_\infty \) that \( f_1(u,t) \in L_u \). By assumption, we have \( \mu_n(du,t) \xrightarrow{\text{weakly}} \mu(du,t) \), and hence

\[
\int_{t_0}^{t_1} dt \int_{R^m} f_1(u,t) \mu_n(du,t) \rightarrow \int_{t_0}^{t_1} dt \int_{R^m} f_1(u,t) \mu(du,t) dt.
\]

However, we have

\[
\int_{t_0}^{t_1} dt \int_{R^m} f_1(u,t) \mu(du,t) = \int_{t_0}^{t_1} h(t) dt \int_{R^m} f(u,t) \mu(du,t),
\]

or \( \int_{t_0}^{t_1} h(t) \psi_n(t) dt \rightarrow \int_{t_0}^{t_1} h(t) \psi(t) dt \) for any \( h(t) \in L_\infty \). This completes the proof of Theorem 3.

**Theorem 4.** Assume that \( f(x,t) \in L_\varphi \) and the function \( f(x,t) \) satisfies the Lipschitz condition with respect to the variable \( x \):

\[
|f(x_1,t) - f(x_2,t)| \leq L(t)|x_1 - x_2|, \quad t \in [t_0, t_1] \quad \text{and} \quad L(t) \in L_1.
\]

The operator \( T: x(t) \mapsto \varphi(t) = c + \int_{t_0}^{t} f(x(r), r) d\tau \) is a contraction mapping with constant \( 1/p \) \((p > 1)\) with respect to the norm

\[
||x|| = \max_{t \in [t_0, t_1]} |x(t)| e^{-p \int_{t_0}^{t} L(t) d\tau}.
\]

**Proof.**

\[
|T(x_1) - T(x_2)| = \left| \int_{t_0}^{t_1} (f(x_1(\tau), \tau) - f(x_2(\tau), \tau)) d\tau \right|
\]

\[
\leq \int_{t_0}^{t_1} |f(x_1(\tau), \tau) - f(x_2(\tau), \tau)| d\tau \leq \int_{t_0}^{t_1} L(\tau)|x_1(\tau) - x_2(\tau)| d\tau
\]

\[
= \int_{t_0}^{t_1} L(\tau)|x_1 - x_2| \exp \left( -p \int_{t_0}^{\tau} L(s) ds \right) \exp \left( p \int_{t_0}^{\tau} L(s) ds \right) d\tau
\]

\[
= \int_{t_0}^{t_1} L(\tau) \left| x_1 - x_2 \right| \exp \left( -p \int_{t_0}^{\tau} L(s) ds \right) \exp \left( p \int_{t_0}^{\tau} L(s) ds \right) d\tau
\]

\[
\leq ||x_1 - x_2|| \int_{t_0}^{t_1} L(\tau) \exp \left( p \int_{t_0}^{\tau} L(s) ds \right) d\tau
\]

\[
= \frac{1}{p} ||x_1 - x_2|| \int_{t_0}^{t} \left( \exp \left( p \int_{t_0}^{\tau} L(s) ds \right) \right) d\left( \int_{t_0}^{\tau} L(s) ds \right) = \frac{1}{p} ||x_1 - x_2|| \exp \left( p \int_{t_0}^{t} L(s) ds \right) d\tau
\]

\[
= \frac{1}{p} ||x_1 - x_2|| \left( \exp \left( p \int_{t_0}^{t} L(s) ds \right) - 1 \right) \leq \frac{1}{p} ||x_1 - x_2|| \exp \left( p \int_{t_0}^{t} L(s) ds \right)
\]

\[
\Rightarrow |T(x_1) - T(x_2)| \leq \frac{1}{p} ||x_1 - x_2|| \exp \left( p \int_{t_0}^{t} L(s) ds \right),
\]

that is,

\[
||T(x_1) - T(x_2)|| \leq \frac{1}{p} ||x_1 - x_2||.
\]

This proves Theorem 4.
§4. PROOF OF ASSERTION 1

Let \( x_n(t) \) be a solution of the system \( S_n \) (6). First we shall prove that

\[
\dot{x}_n(t) \xrightarrow{L_\infty\text{-weakly}} \int_{\mathbb{R}^m} f(x^0, u, t) \mu(du, t).
\]

By assumption we have

\[
\dot{x}_n(t) = \int_{\mathbb{R}^m} f(x_n, u, t) \mu_n(du, t).
\]

However,

\[
\int_{\mathbb{R}^m} f(x_n, u, t) \mu_n(du, t) = \int_{\mathbb{R}^k} E_{x_n}(dx) \int_{\mathbb{R}^m} f(x, u, t) \mu_n(du, t),
\]

or

\[
\dot{x}_n(t) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^m} f(x, u, t) \tilde{\mu}_n(dx, du, t),
\]

where \( \tilde{\mu}_n(dx, du, t) = E_{x_n}(dx) \times \mu_n(du, t) \).

Now it follows from \( x_n \xrightarrow{\text{w}} x^0 \) that \( E_{x_n}(dx) \xrightarrow{\text{w}} E_{x^0}(dx) \). By assumption, we have \( \mu_n(du, t) \rightarrow \mu(du, t) \), and hence, as is known, \( \tilde{\mu}_n(du, t) \xrightarrow{\text{w}} \tilde{\mu}(du, t) \), where \( \tilde{\mu}(du, t) = E_{x^0}(dx) \times \mu(du, t) \). By Theorem 3, we have

\[
\int_{\mathbb{R}^k} \int_{\mathbb{R}^m} f(x, u, t) \tilde{\mu}_n(dx, du, t) \xrightarrow{L_\infty\text{-weakly}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^m} f(x, u, t) \tilde{\mu}(dx, du, t).
\]

However,

\[
\int_{\mathbb{R}^k} \int_{\mathbb{R}^m} f(x, u, t) \tilde{\mu}(dx, du, t) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^m} f(x, u, t) E_{x^0}(dx) \mu(du, t) = \int_{\mathbb{R}^m} f(x^0, u, t) \mu(du, t),
\]

that is,

\[
\dot{x}_n \xrightarrow{L_\infty\text{-weakly}} \int_{\mathbb{R}^m} f(x^0, u, t) \mu(du, t).
\]

Let us show that \( x_n(t) \rightarrow x^0(t) \xrightarrow{\text{w}} \dot{x}_n(t) \rightarrow \dot{x}^0(t) \). Assume that \( \dot{x}_n(t) \rightarrow \xi(t) \). Consider the expression

\[
x^0(t_0) + \int_{t_0}^{t_1} \xi(t) \, dt = x^0(t_0) + \int_{t_0}^{t_1} \chi_{[t_0, t_1]} \xi(t) \, dt = x^0(t_0) + \lim_{n \rightarrow \infty} \int_{t_0}^{t_1} \dot{x}_n(t) \, dt = x^0(t_0) + \lim_{n \rightarrow \infty} \int_{t_0}^{t_1} \dot{x}_n(t) \, dt = x^0(t_0) + \lim_{n \rightarrow \infty} (x_n(t_*) - x_n(t_0)) = x^0(t_*),
\]

and thus we obtain \( x^0(t_0) + \int_{t_0}^{t_1} \xi(t) \, dt = x^0(t_*) \), which means that \( \dot{x}^0(t) (=) \xi(t) \); \( x^0(t_0) = x^0 \) (here and henceforth, the symbols (=) and (≤) stand for the equality and inequality "almost everywhere"). Since the limit of the sequence \( \{\dot{x}_n\} \) is unique, the function \( x^0(t) \) satisfies the relation \( \dot{x}^0(t) = \int_{\mathbb{R}^m} f(x^0, u, t) \mu(du, t) \). This completes the proof of Assertion 1.


§5. PROOF OF ASSERTION 2

Let $x^0(t)$ and $x_n(t)$ satisfy the systems $S$ and $S_n$, respectively, with initial data $x^0(t_0) = x^0_0$ and $x_n(t_0) = \xi_n$. Then the systems $S$ (7) and $S_n$ (6) together with these initial data are equivalent to the following systems of integral equations:

$$x_n(t) = \xi_n + \int_{t_0}^t \int_{\mathbb{R}^m} f(x_n(\tau), u, \tau) \mu_n(du, \tau) d\tau,$$

$$x^0(t) = x^0_0 + \int_{t_0}^t \int_{\mathbb{R}^m} f(x^0(\tau), u, \tau) \mu(du, \tau) d\tau.$$

Substitute a solution $x^0(t)$ of the system (9) into (8). Since $x^0(t)$ is not a solution of the system (8), it follows that the following relation holds:

$$x^0(t) = x^0_0 + \int_{t_0}^t \int_{\mathbb{R}^m} f(x^0(\tau), u, \tau) \mu(du, \tau) d\tau + Q_n(t).$$

Let us show that $Q_n(t) \to 0$. On subtracting (9) from (10), we obtain

$$Q_n(t) = \int_{t_0}^t \int_{\mathbb{R}^m} f(x^0(\tau), u, \tau) (\mu(du, \tau) - \mu_n(du, \tau)) d\tau.$$

For any chosen $t_* \in [t_0, t_1]$ we have $Q_n(t_*) \to 0$ because

$$Q_n(t_*) = \int_{t_0}^{t_*} \int_{\mathbb{R}^m} f(x^0(\tau), u, \tau) (\mu(du, \tau) - \mu_n(du, \tau)) d\tau$$

$$= \int_{t_0}^{t_1} \int_{\mathbb{R}^m} \chi_{[t_0, t_*]} f(x^0(\tau), u, \tau) (\mu(du, \tau) - \mu_n(du, \tau)) d\tau$$

$$= \int_{t_0}^{t_1} \int_{\mathbb{R}^m} f_1(x^0(\tau), u, \tau) (\mu(du, \tau) - \mu_n(du, \tau)) d\tau,$$

where $f_1(x^0(t), u, t) = \chi_{[t_0, t_*]}(t) f(x^0(u, t))$ is an element of $L_{xu}$. By assumption we have

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^m} f_1(x^0(\tau), u, \tau) (\mu(du, \tau) - \mu_n(du, \tau)) d\tau \to 0.$$

Let us prove that the convergence $Q_n(t) \to 0$ is uniform. We have

$$|Q_n(t)| \leq \int_{t_0}^t \int_{\mathbb{R}^m} \max_{u \in K_u} |f(x^0(\tau), u, \tau)| (\mu(du, \tau) - \mu_n(du, \tau)) d\tau \leq \int_{t_0}^t \varphi(\tau) d\tau,$$

where $\varphi(t) = 2 \max_{u \in K_u} |f(x^0(t), u, t)|$, and

$$|\Delta Q_n| = |Q_n(t'') - Q_n(t')| \leq \int_{t'}^{t''} \varphi(\tau) d\tau \quad \forall t', t'' \implies |Q_n(t)| (\leq) \varphi(t),$$

and it follows from the relation $Q_n(t) \to 0$, $t \in [t_0, t_1]$, that $Q_n(t) \to 0$. 

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Consider the operator $A_n : C[t_0, t_1] \to C[t_0, t_1]$ defined by

$$A_n x(t) = x(t) - x_0 - \int_{t_0}^t \int_{\mathbb{R}^m} f(x(\tau), u, \tau) \mu_n(du, \tau) d\tau.$$ 

It follows from (10) that $A_n x^0(t) = Q_n(t)$. The operator $A_n$ can be written in the form $A_n = E + T_n$. The operator $E : x^0(t) \to x^0(t)$ covers with the constant 1 with respect to an arbitrary norm. The operator

$$T_n : x^0(t) \to -x_0 - \int_{t_0}^t \int_{\mathbb{R}^m} f(x^0(t), u, t) \mu_n(du, t) dt$$

is a contraction mapping, with the constant $1/p$, with respect to the norm

$$||x|| = \max_{t \in [t_0, t_1]} \left| x \exp \left( -p \int_{t_0}^t L(t) dt \right) \right|$$

in a neighborhood of the point $x^0(t)$. Indeed, consider the ball $B_\rho(x^0) = \{ x(t) : ||x^0(t) - x(t)|| \leq \rho \}$, where $\rho = \max_{t \in [t_0, t_1]} ||x^0(t)||$. The ball $B_\rho(x^0)$ is compact in $\mathbb{R}^n$. Consider the compact set $K_{xu} = B_\rho(x^0) \times K_u$. The function $f(x, u, t)$ satisfies the Lipschitz condition on $K_{xu}$ (condition 2) of §2). Then the function $F_n(x, t) = \int_{\mathbb{R}^m} f(x, u, t) \mu_n(du, t)$ satisfies the Lipschitz condition with respect to the variable $x$ with the same constant. Then it follows from Theorem 4 that the operator $T_n$ is a contraction mapping with the constant $1/p$. By the theorem on coverings [4], the operator $A_n$ covers with the constant $a = 1 - 1/p$. Since $A_n(B_{||Q_n||/a}(x^0)) \supset B_{||Q_n||}(Q_n)$ and $0 \in B_{||Q_n||}(Q_n)$, it follows that there exists a sequence $x_n(t) : ||x_n(t) - x^0(t)|| \leq ||Q_n||/a$ such that $x_n(t)$ vanishes ($x_n \to 0$), i.e., $x_n(t)$ is a solution of the system (8). Since $Q_n \to 0$, we have $||x_n(t) - x^0(t)|| \to 0$. This proves Assertion 2.

It follows from Assertions 1 and 2 proved above that the system $S_n$ (6) tends to the system $S$ (7) as $n \to \infty$.

§6. CONCLUSIONS

Now we can apply the above theory to obtain the answer to the problem posed in the introduction. We establish the following assertions.

Theorem 5. Let $f(x, u, t) \in L_{xu}$, let $u_n \overset{L_\infty-\text{weakly}}{\to} 0$, and let $E_{u_n}(du, t) \overset{\text{weakly}}{\to} \mu(du, t)$. Then the system (1) passes to the system (2), and relation (3) is satisfied.

Proof. The system (1) is equivalent to the system

$$\dot{x}(t) = \int_{\mathbb{R}^m} f(x, u, t) \mu_n(du, t),$$

(11)

where $\mu_n(du, t) = E_{u_n} = \delta(u - u_n(t))du$, i.e., the system (11) is a system of the type $S_n$ (6). The proof of the theorem follows from Assertions 1 and 2. Since $u_n \overset{L_\infty-\text{weakly}}{\to} 0$, it follows from Theorem 1 that condition (3) is satisfied. This proves Theorem 5.

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Theorem 6. Let \( \mu(du,t) \in V_{K_u} \) and let the condition \( \int u\mu(du,t) = 0 \) hold. Then there exists a sequence \( u_n \xrightarrow{L_\infty \text{-weakly}} 0 \) such that the system (2) is a limit system for the system (1).

Proof. Since the set \( V^0_{K_u} \) is weakly dense, it follows that, for each measure \( \mu \in V_{K_u} \), there exists a sequence of measures \( E_n \) such that \( E_n \xrightarrow{\text{weakly}} \mu(du,t) \). We write \( E_n(du,t) = \mu_n(du,t) \). Since the condition (3) is satisfied, it follows from Theorem 1 that \( u_n \xrightarrow{L_\infty \text{-weakly}} 0 \). Then, by Theorem 5, the system (1) converges to the system (2). This completes the proof of Theorem 6.

Thus, Theorems 5 and 6 give the complete description of the class of limit systems, and this class does not contain the limiting sequence \( u_n \xrightarrow{L_\infty \text{-weakly}} 0 \). The measure \( \mu \in V_{K_u} \) is a parameter of this class.

These theorems allow us to pass to the limit for more complicated systems than that of the form (1). We write \( \dot{a}(t) = u_n(t) \) and \( a(t) = \int u_n(\tau) \, d\tau \) and consider the following system:

\[
\begin{align*}
\dot{x} &= f(x,y,a,\dot{a},t), \\
\dot{y} &= F(x,y,a,\dot{a},t) + \dot{a}\varphi(x,a,t),
\end{align*}
\]  

(12)

where the functions \( f \) and \( F \) are continuous with respect to \( x, y, a, \) and \( \dot{a} \) for any \( t \in [t_0,t_1] \), measurable with respect to \( t \), and satisfy the Lipschitz condition with respect to the variables \( x \) and \( y \) on the compact set \( K_{xy} \times B_\delta(a) \); finally, \( \varphi \) is a smooth function. Systems of the form (12) often occur in vibration problems in mechanics (for instance, if the force field is vibrating). Let us write the limit system as \( a \xrightarrow{\text{weakly}} 0 \) and \( \dot{a} \xrightarrow{C} 0 \). Applying the transformation

\[
y = z + \dot{a}\varphi
\]  

(13)

we can present the system (12) in the form

\[
\begin{align*}
\dot{x} &= f(x,z + \dot{a}\varphi,a,\dot{a},t), \\
\dot{z} &= F(x,z + \dot{a}\varphi,a,\dot{a},t) - \dot{a}\varphi'_z(x,a,t)f(x,z + \dot{a}\varphi,a,\dot{a},t) - \dot{a}\varphi'_a(x,a,t) - \dot{a}^2\varphi'_a(x,a,t).
\end{align*}
\]  

(14)

Taking account of condition (3), we see that the limit values of \( z \) and \( y \) coincide (13). By Theorem 5, the limit system becomes

\[
\begin{align*}
\dot{x} &= \int_{\mathbb{R}^m} f(x,y + u\varphi,0,u,t)\mu(du,t), \\
\dot{z} &= \int_{\mathbb{R}^m} F(x,y + u\varphi,0,u,t)\mu(du,t) - \int_{\mathbb{R}^m} u\varphi'_z(x,0,t)f(x,y + u\varphi,0,u,t)\mu(du,t) \\
&- \dot{a}^2\varphi'_z(x,0,t), \quad (\dot{a}^2 = \int_{\mathbb{R}^m} u^2\mu(du,t)),
\end{align*}
\]  

(15)

where \( \mu(du,t) \) is the weak limit of the measures \( E_\dot{a}(du,t) \). By Theorem 6, for each measure \( \mu(du,t) \in V_{K_u} \) satisfying condition (3) we can reconstruct the vibration \( \dot{a} \xrightarrow{\text{weakly}} 0 \). These functions must be differentiable for the system (12) to make sense. If the compact set \( K_{xy} \) has the property that the smooth functions with values in this compact set are dense in the set of all functions with values in \( K_{xy} \) (for instance, if \( K_{xy} \) is convex), then we can choose a sequence of smooth functions \( \dot{a} \xrightarrow{\text{weakly}} 0 \) close to \( \dot{a}(t) \) with respect to the \( L_1 \) norm. Then it follows from Theorem 2 that \( \mu(du,t) \) is the weak limit of the measures \( E_{\dot{a}}(du,t) \).
§7. EXAMPLES

Before proceeding with the examples, we present several useful formulas and assertions that allow us to avoid cumbersome calculations when solving the stabilization problems for systems and writing out the Hamiltonian functions for systems with vibrating constraints in a vibrating potential field.

Assume that, for a system of $N$ mass points, the Lagrange function has the form

$$L(x, \dot{x}) = \frac{1}{2}(T\dot{x}, \dot{x}) + p(x)x + v(x), \quad (16)$$

where $x$ are Cartesian coordinates of the system, $x \in \mathbb{R}^n$ ($n = N, 2N, 3N$), and $T$ is an $n \times n$ matrix.

Assume that the equation $L'_{\dot{x}} = 0$ is solvable with respect to $\dot{x}$. Then the Hamiltonian function has the form

$$H(x, \psi) = \frac{1}{2}(T^{-1}(\psi - p), (\psi - p)) - v(x). \quad (17)$$

If a constraint $x = X(y)$ is imposed on the system ($y$ are generalized coordinates of the system of $N$ points, $y \in \mathbb{R}^m$), then the corresponding Hamiltonian function becomes

$$H(y, \psi) = \frac{1}{2}(X'^TXX'_y)^{-1}(\psi - pX'_y), (\psi - pX'_y)) - v(X(y)), \quad (18)$$

where $X'_y$ is an $n \times m$ matrix.

1) $L(x, \dot{x}) = \frac{1}{2}(T\dot{x}, \dot{x}) + p(x)x + v(x) \quad (19)$

If a vibrating constraint

$$x = X(y) + a(t)R(y) \quad (20)$$

is imposed on the system ($y \in \mathbb{R}^m$, $a(t)$ is the vibration, $a(t) \in \mathbb{R}^1$, $a(t)$ is small ($a \to 0$), and $|a(t)|$ is bounded ($a \to 0$)), then we have

$$H(y, \psi, a, \dot{a}) = \frac{1}{2}(S(\psi - B^*TR\dot{a}), (\psi - B^*TR\dot{a})) - \frac{\dot{a}^2}{2}(TR, R) - w(y, a), \quad (21)$$

(with $S = (B^*TB)^{-1}$, $B = X'_y + aR'_y$, and $w(y, a) = v(X(y) + R(y)a))$.

2) $L(x, \dot{x}) = \frac{1}{2}(T\dot{x}, \dot{x}) + v(x), \quad (22)$

3) $L(x, \dot{x}) = \frac{1}{2}(T\dot{x}, \dot{x}) + p(x)x + v(x), \quad (23)$

$$H(y, \psi, a, \dot{a}) = \frac{1}{2}(S(\psi - B^*TR\dot{a} - B^*p), (\psi - B^*TR\dot{a} - B^*p))$$

$$- \frac{\dot{a}^2}{2}(TR, R) - pR\dot{a} - w(y, a). \quad (24)$$

Assertion 3. Let a Hamiltonian function $H(x, \psi)$ be given. If at an equilibrium point we have $H''_{\psi\psi} > 0$ (the Hessian is positive definite), then by adding a function $C\delta H(x) \quad (C = \text{const})$ such that $\delta H''_{xx} |_{EP} > 0$ we can make the function $H + C\delta H$ have a positive-definite second form at the equilibrium point by choosing the constant $C$. 

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Assertion 4. Given a Hamiltonian system with function $\mathcal{H}(x, \psi)$, consider $\mathcal{H}_1 = \mathcal{H}(x, \psi) + aU(x, a)$. After the change of variables $\psi = z - aU_x'$ we obtain the system with the function

$$\mathcal{H}_1(x, z, a, a) = \mathcal{H}(x, z - aU_x', a, a) + a^2U_a' \tag{25}$$

with respect to the variables $x, z$, and to the limit value of $\mathcal{H}_1$ (for small $a (a \rightarrow 0)$ and bounded $|\dot{a}| (\dot{a} \rightarrow 0)$), an action corresponds. Note that the limit values of $z$ and $\psi$ coincide by (3) ($\tilde{z} = \tilde{\psi}$).

For a known Hamiltonian function in the case of vibrating constraint (21) and (24) and a vibrating field (25), we must write out the limit system of equations

$$\dot{y} = \mathcal{H}'_1(y, \psi, a, a), \quad -\dot{\psi} = \mathcal{H}'_1(y, \psi, a, a) \tag{26}$$

(the bar denotes the limit as $a \rightarrow 0$ and $\dot{a} \rightarrow 0$). According to the above theory, this limit system has the form

$$\dot{y} = \mathcal{H}'_1(y, \psi, a, a) = \int \mathcal{H}'_1(y, \psi, 0, u)\mu(du, t),$$

$$-\dot{\psi} = \mathcal{H}'_1(y, \psi, a, a) = \int \mathcal{H}'_1(y, \psi, 0, u)\mu(du, t), \tag{27}$$

where $\mu(du, t)$ is the weak limit of the measures $E_a(du, t)$ and condition (3) is satisfied. We can prove the following general assertion:

$$\int F'_x(z, u)\mu(du) = \left( \int F(x, u)\mu(du) \right)'_x$$

provided that $F(x, u)$ and $F'_x(x, u)$ are continuous functions of their arguments. Therefore, the limit Hamiltonian function corresponds to the limit system. Let us write out the limit functions in (21) and (24), taking into account the fact that the terms with $a(t)$ and $\dot{a}(t)$ vanish (because $a \rightarrow 0$ and according to formula (3) of the introduction):

$$\mathcal{H}_1(y, \psi, a, a) = \frac{1}{2}(\dot{S}\psi, \psi) - w(y, 0) + \frac{a^2}{2}(\dot{S}X_{y'}^*TR, X_{y'}^*TR) - \frac{a^2}{2}(TR, R), \tag{28}$$

$$\left( \dot{S} = (X_{y'}^*TX_{y'}')^{-1}, \quad w(y, 0) = v(X(y)), \quad \dot{a}^2 = \int u^2\mu(du, t) \right),$$

$$\mathcal{H}_1(y, \psi, a, a) = \frac{1}{2}(\dot{S}\psi, \psi) - (\dot{S}X_{y'}^*p, \psi) + \frac{1}{2}(\dot{S}X_{y'}^*p, X_{y'}^*p) - w(y, 0) + \frac{\dot{a}^2}{2}(S_{y'}^*TR, X_{y'}^*TR) - \frac{\dot{a}^2}{2}(TR, R) \tag{29}$$

Note that $\mathcal{H}_1(y, \psi, a, a) = \mathcal{H}(y, \psi) + \dot{a}^2\delta \mathcal{H}(y, a)$, where $\mathcal{H}(y, \psi)$ is the Hamiltonian function before the vibration. If the question is whether the equilibrium position can be stabilized, then we must verify several conditions. For an equilibrium point to remain an equilibrium point after vibration it is necessary to have $(\delta \mathcal{H})'_y = 0$. In the case of (25), at the equilibrium point (EP), the potential field must satisfy the condition $U'_x|_{EP} = 0$. Since, as a result of the vibration, the additional term in (28) and (29) does not depend on $\psi$, according to
Assertion 3, it suffices to clarify whether the matrix $(\delta \mathcal{H})''_{yy}$ is positive definite at the equilibrium point.

I. Let us illustrate the application of the above formulas by the classical example of stabilization of the inverted pendulum.

Consider a plane pendulum (Fig. 1) with vertically vibrating point of support (it is assumed that $l = 1$).

![Fig. 1.](image)

The Lagrange function has the form $L(x, \dot{x}) = \dot{x}^2/2 - U(x)$ (where $x = (x_1, y_1)$ are the Cartesian coordinates of the point $A$ and $U(x)$ is the potential energy), that is, the Lagrange function has the form (19), where $T = I$ (the identity matrix) and $U = py_1$. Taking account of the vibrating constraint, we write $x_1 = \cos \varphi$ and $y_1 = \sin \varphi + a(t)$, i.e., $x = X(\varphi) + Ra$, where $X = (\cos \varphi, \sin \varphi)$. $R = (0^T)$, $a(t)$ is a scalar, and $B = X_\varphi' = (-\sin \varphi)$. Since $B'R_{1/2} = 0$, the stabilization is possible in principle. Applying formula (28), we obtain the limit Hamiltonian function

$$\overline{\mathcal{H}}_1(\varphi, \psi, a, \dot{a}) = (1/2)\dot{\psi}^2 + \psi \sin \varphi + (c^2/2) \cos^2 \varphi - c^2/2$$

(with $T = I$, $\overline{S} = (X_\varphi' X_\varphi'\psi)^{-1} = I$, $(R, R) = 1$, and $c^2 = \dot{a}^2$).

Here and in the subsequent examples, it suffices to choose a measure not depending on $t$ ($\mu(du, t) = \mu(du)$). Then we have $\dot{a}^2 = \text{const} > 0$ and $\delta \mathcal{H}(\varphi) = (c^2/2) \cos^2 \varphi - c^2/2$. Since $(\delta \mathcal{H})''_{\varphi \varphi} = -c^2 \cos 2\varphi = c^2 > 0$, it follows from Assertion 3 that by choosing $c^2$ we can transform the point of unstable equilibrium into a stable one.

II. Consider an $N$-link pendulum (which is assumed to be plane with $l_i = 1$, $i = 1, 2, \ldots, N$):

$$L = \frac{1}{2} \dot{x}^2 - U(x) = \frac{1}{2} \sum_{k=1}^N (\dot{x}_k^2 + \dot{y}_k^2) - p \sum_{k=1}^N y_k$$

(where $(x_k, y_k)$ are the Cartesian coordinates of the $k$th mass point). The Lagrange function has the form (19). Taking account of the vibrating constraint, we obtain $x_k = \sum_{i=1}^k \cos \varphi_i$, $y_k = \sum_{i=1}^k \sin \varphi_i + a(t)$, i.e., a constraint of the type (20), where $R = (0, 1, \ldots, 0, 1)$, $R \in \mathbb{R}^{2N}$, and

$$B = X_\varphi' = \begin{pmatrix}
-\sin \varphi_1 & 0 & 0 & \ldots & 0 \\
\cos \varphi_1 & 0 & 0 & \ldots & 0 \\
-\sin \varphi_1 & -\sin \varphi_2 & 0 & \ldots & 0 \\
\cos \varphi_1 & \cos \varphi_2 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
-\sin \varphi_1 & -\sin \varphi_2 & \ldots & \ldots & -\sin \varphi_N \\
-\cos \varphi_1 & \cos \varphi_2 & \ldots & \ldots & \cos \varphi_N \\
\end{pmatrix}.$$
so that $B$ is a $2N \times N$-matrix. Since $B^* R|_{EP} = (N \cos \varphi_1, (N-1) \cos \varphi_2, \ldots, \cos \varphi_N)|_{EP} = 0$, it follows that the equilibrium point remains an equilibrium point. Applying formula (28), we obtain

\[
\mathcal{H}_1(\varphi, \psi, a, \dot{a}) = \frac{1}{2} (\ddot{\psi} \dot{\varphi} + \sum_{k=1}^N (N - k + 1) \sin \varphi_k + c^2 \left(\dot{X}_\varphi^* R, X'_\varphi R\right) - \frac{c^2}{2} |R|^2)
\]

\[
\left(\hat{S} = (X'_\varphi X'_\varphi)^{-1}, \ |R|^2 = N, \ \psi \in \mathbb{R}^{2N}, \ c^2 = \hat{a}^2)\right).
\]

Let us prove that, the matrix $(\delta \mathcal{H})_\varphi^\prime$ is positive definite at the equilibrium point. We have $\delta \mathcal{H}(\varphi) = (c^2/2) (\dot{S} X'_\varphi R, X'_\varphi R) - \left(\frac{c^2}{2}\right) |R|^2$. Since $X'_\varphi R|_{EP} = 0$, it follows that $(\delta \mathcal{H})_\varphi^\prime|_{EP} = ((X'_\varphi R)_\varphi^*) (X'_\varphi X'_\varphi)^{-1} (X'_\varphi R)_\varphi^*|_{EP}$. However, a matrix of this type is always nonnegative definite, and since $|((X'_\varphi R)_\varphi^*)|_{EP} = 0$, it follows that $(\delta \mathcal{H})_\varphi^\prime|_{EP}$ is positive definite. Therefore, by Assertion 3, we can choose $c^2 = \hat{a}^2$ so that the point of unstable equilibrium becomes stable.

III. Let us present an example of lifting a rod by means of a vibration from any acute position to a vertical one.

Consider the following construction: on a rigid weightless rod of length $l = 1$, a vibrating load is fastened $(p = 1)$. The endpoints of the rod can freely move along two mutually perpendicular lines.

Choose a system of coordinates as is shown in Fig. 2. We place the vibrating load at a point $A$ at a distance $\delta$ from the lower endpoint of the rod.

In the Cartesian coordinates, the Lagrange function has the form $L = (1/2) \dot{x}^2 - U(x)$, $x \in \mathbb{R}^2$, where $U(x)$ is the potential energy of the system.

The vibrating constraint has the form (20): $x = X(\varphi) + a R(\varphi)$, where

\[
X(\varphi) = \begin{pmatrix} (1 - b) \cos \varphi \\ b \sin \varphi \end{pmatrix}, \quad R(\varphi) = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix}.
\]

Since $B = X'_\varphi + a R'_\varphi$, it follows that $B^* R|_{\pi/2} = -\cos \varphi \sin \varphi|_{\pi/2} = 0$, i.e., the equilibrium point remains an equilibrium point. Applying formula (28) we obtain

\[
\mathcal{H}_1(\varphi, \psi, a, \dot{a}) = \frac{1}{2} \left( \frac{\psi^2}{(1 - b)^2 \sin^2 \varphi + b^2 \cos^2 \varphi} + b \sin \varphi + \frac{c^2 \hat{a}^2}{2} \right) \left(\sin^2 \varphi \cos^2 \varphi + \frac{b^2 \cos^2 \varphi}{(1 - b)^2} \right) - \frac{1}{2} \hat{a}^2 R^2 \ (R^2 = 1),
\]

\[
\delta \mathcal{H}(\varphi) = \frac{c^2}{2} \sin^2 \varphi \left(1 - \sin^2 \varphi\right) - \frac{c^2}{2} (c^2 = \hat{a}^2).
\]
As proved above, we can make the point $\varphi = \pi/2$ a point of stable equilibrium. The problem of interest is as follows: what is the lowest position from which the motion can start? Assume that we start from a position $(\varphi_0, \psi_0)$, and let $b$ be chosen. Since the Hamiltonian remains constant along the motion, and the first term in the expression (25) for $\mathcal{H}_1$ increases, it follows that the product $\delta \mathcal{H}(\varphi)$ must decrease, i.e., the condition $(\delta \mathcal{H})'_\varphi < 0$ is necessary. However,

$$(\delta \mathcal{H})'_\varphi = \left( \frac{c^2 \sin^2 \varphi (1 - \sin^2 \varphi)}{2 (1 - 2b^2) \sin^2 \varphi + b^2} \right)' = F(\sin^2 \varphi)$$

is a function of $\sin^2 \varphi$. On investigating this function for all values of the parameter $b$, we obtain the dependence of the angle from which the motion can start on the position of the vibrator on the rod:

$$b < \sin^2 \varphi < 1.$$  \(31\)

The most advantageous situation is related to the least possible value of the angle (motion from the horizontal position). However, since condition (31) must hold, it follows that the vibrator must be placed in a low position, i.e., $b \sim \varepsilon$. To obtain the effect of motion for a finite rate of vibration ($c^2 < \infty$) we must choose an angle for which $\sin^2 \varphi \sim k \varepsilon$. This follows from the condition

$$\left| (\delta \mathcal{H})'_\varphi \right| > b \cos \varphi, \quad \text{i.e.,} \quad \frac{1}{2} a^2 > \left| \frac{b \cos \varphi ((a - 2b) \sin^2 \varphi + b^2)}{(2b - 1) \sin^4 \varphi - 2b \sin^2 \varphi + b^2} \right|.$$  

IV. The following example shows that the vibration can perform the “flight” of a rod from the horizontal position.

Let the mass of the rod be concentrated at its endpoints, and denote their coordinates by $p, r \in \mathbb{R}^3$ (at the initial instant we have $p = (p_1, p_2, 0)$ and $r = (r_1, r_2, 0)$). At the center of the rod, whose coordinate will be denoted by $q = (q_1, q_2, q_3)$, we place a vibrational load (at the initial instant we have $q_3 = 0$). Denote the coordinates of the position of the vibrator by $w$. Assume that the masses of all three mass points are equal to one. The Lagrange function is written in the form (19): $L = (1/2) \dot{x}^2 - U(x)$, where $x = (p, r, w)$. Introduce a new vector variable $s$ and take account of the vibrating constraint and the fact that the rod is inextensible; then we obtain $p = q + s, r = q - s$, and $w = q + a(t)n(q)$ ($a$ is a scalar and $n \in \mathbb{R}^3$). Let us express this constraint in the form (20) as follows: $x = X(y) + aR(y)$, where $y = (q, s)$. Then we have

$$X(q, s) = \begin{pmatrix} q + s \\ q - s \\ s \end{pmatrix}; \quad R = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad X'_y = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad X'_y X'_y = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix};$$

$$S = (X'_y X'_y)^{-1} \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}.$$  

Denote by $\rho$ the vector $(0, 0, 1)$. Then the potential energy has the form $U(x) = (g \rho, 3q + an(q))$. Let us apply formula (28): $\mathcal{H}(y, \psi, a, \vec{a}) = (1/2)(S\psi, \psi) + (g \rho, 3q) - (c^2/3)(n, n), \quad \psi = (\psi^0, \psi^1)$. Write out the motion relations:

$$\dot{q} = (1/3)\psi^0, \quad \dot{\psi}^0 = 3g \rho - (c^2/3)(n, n)'_q, \quad -\dot{\psi}^1 = 0.$$  

This implies

$$\dot{q} = (1/9)c^2(n, n)'_q - g \rho.$$  \(32\)

Let us choose $n(q)$ so that the first summand of (32) has the direction of the vector $\rho = (0, 0, 1)$, and then by choosing $c^2$, we can obtain the relation $\dot{q} > 0$, i.e., we shall have a vertical lifting. For the inner product $(n, n)$ not to depend on $q_1$ and $q_2$, it suffices to take the vector $n(q)$ in the form $(0, 0, q_3 + \text{const})$. We must add the constant indeed, because $q_3 = 0$ at the initial instant.
PART II. DISCONTINUOUS CASE

INTRODUCTION

In this part of the paper we treat the case in which the right-hand side of the system (1) is discontinuous with respect to the phase variable. This class of problems includes, for example, problems in mechanics with dry friction. This leads to the consideration of convex differential inclusions, with compact graph, of the form

\[ \dot{z}(t) (\in) \tilde{f}(x, u_n, t), \quad u_n(t) \xrightarrow{\text{weakly}} 0. \]  

(33)

In this paper we prove a theorem that allows us to describe the entire class of limit differential inclusions.

First we discuss some approaches for obtaining differential inclusions from a differential equation whose right-hand side is discontinuous with respect to the phase variable, i.e., from the equation

\[ \dot{z}(t) = f(x, t). \]  

(34)

In the paper [5], the function \( \tilde{f}(x, t) \) is defined as follows:

\[ \tilde{f}(x, t) = \text{conv} \uparrow \lim_{x' \to x} f(x', t) \]  

(35)

(conv stands for the convex closure and \( \uparrow \lim \) for the upper topological limit). This definition is invariant with respect to a smooth change of variables \( z = X(y, t) \), i.e., we have the following commutative diagram:

\[
\begin{array}{ccc}
 f(x, t) & \longrightarrow & \tilde{f}(x, t) \\
 \downarrow_{z=X(y, t)} & & \downarrow_{z=X(y, t)} \\
 \varphi(y, t) = f(X(y, t), t) & \longrightarrow & \tilde{\varphi}(y, t)
\end{array}
\]  

(36)

Thus we can consider, along with Eq. (34), the equation \( \dot{z} = f(x, t) + \varphi(t) \) for all \( \varphi(t) \). Then by setting \( x - \varphi = y \), we obtain

\[ \dot{y} = f(y + \varphi, t). \]  

(37)

For problems in mechanics with dry friction, this change of variables is used to take account of vibration.

Definition (35) was criticized. Reference [6] cites an example from automatic control theory in which this approach leads to the same inclusion in two cases that technically differ by means of different relays.

The authors of [6] propose another approach, namely, to consider equations of the form

\[ \dot{z} = F(x, y, t), \]  

(38)

where \( F \) is a continuous function of its arguments and \( y = u(x, t) \) is a discontinuous "kernel." Then the authors likewise obtain the differential inclusion

\[ \dot{z} \in \text{conv} F(x, \text{conv} \uparrow \lim_{x' \to x} u(x', t), t) \]  

(39)
however, the set $\tilde{f}(x,t) = \text{conv} F(x,\text{conv} \{x' + xu(x',t),t\})$ turns out to be broader than in (35). The different results in the two approaches of [5, 6] in the above example are related to the fact that, in contrast to the invariance of the operators of changing the variables and of taking the convex hull, the operations of projecting to a subspace and taking the convex hull do not commute. Consider the following example:

$$\dot{x} = s_1(x) + b s_2(y), \quad \dot{y} = s_1(x) + b s_2(y), \quad (40)$$

where $s_1$ and $s_2$ are discontinuous functions. Let us take the subspace $x = y$; then we obtain the inclusion

$$\dot{x}(t) \in \tilde{f}(x), \quad (41)$$

where $f(x) = s_1(x) + b s_2(x)$. At the same time, if we first apply the operation of taking the convex hull to (40) and set $x = y$ after it, then we can obtain an inclusion that differs from (41) (for instance, this is always the case if $s_1 = -s_2$).

To include a broader class of motions, we can always mean by a differential inclusion a pair of the form (33), (35), but we must consider a broader system than (34), apply to it the operation of taking the convex hull, and then take the restriction to the subspace we need.

In what follows we do not discuss the way in which we obtain a differential inclusion and treat inclusions of the form (33). To prove the main theorem on the passage to the limit in (33) as $u_n \rightarrow 0$, we need an auxiliary technique.

§1. AUXILIARY TECHNIQUE

The results of this section essentially exploit the notions of Carathéodory integrand and normal integrand. Let us present the necessary definitions.

Let $X$ be a compact set and let $t \in [t_0, t_1]$.

**Definition 1.** A function $f(x,t)$ is called a *Carathéodory integrand* if it is finite and continuous with respect to $x$ for any $t \in [t_0, t_1]$ and is measurable with respect to $t$ for any $x \in X$.

**Definition 2.** A function $f(x,t)$ is called a *normal integrand* if

$$f(x,t) = \lim_{n \rightarrow \infty} \inf f_n(x,t) = \inf f_n(x,t) \quad (42)$$

or

$$f(x,t) = \lim_{n \rightarrow \infty} \sup f_n(x,t) = \sup f_n(x,t), \quad (43)$$

where $f_n(x,t)$ are Carathéodory integrands and the arrows $\downarrow$ and $\uparrow$ stand for monotone increasing (monotone decreasing) passage to the limit.

**Remark.** This definition of normal integrand is not traditional. For our case, in which $X$ is a compact set, this definition is equivalent to the standard one by the Levin theorem [7].

The theorem of [7] presents necessary and sufficient conditions for the function $f(x,t)$ that is upper (lower) semicontinuous with respect to $x$ for each $t \in [t_0, t_1]$ and measurable with respect to $t$ for each $x \in X$ to be a normal integrand, namely, these conditions are that the mappings

$$t \rightarrow \max_X (f(x,t) - \varphi(x)) \quad \text{and} \quad t \rightarrow \min_X (f(x,t) - \varphi(x)), \quad (44)$$

respectively, be measurable for any $\varphi(x) \in C(X)$.

Now let us pass to the exposition of notions and assertions we need in the proof of the main theorem on the passage to the limit in (33).

Assume that on a compact set $K_y$, a convex multivalued mapping $y \rightarrow Q(y) \in \mathbb{R}^d(x)$ with compact graph is given, where $\sigma(dy)$ is a unit positive Radon measure.
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Definition 3.
\[ Z = \int_{K_y} Q(y)\sigma(dy) \overset{\text{def}}{=} \bigcup_{\xi \in Q(y)} \int_{K_y} \xi(y)\sigma(dy), \]  
where \( \xi(y) \) are all measurable samples with respect to the measure \( \sigma \).

Lemma 1. \( Z \) is a compact convex set.

Proof. Since \( Q(y) \) is a convex mapping with compact graph, we can readily show that \( Z \in \mathbb{R}^d \) is a bounded convex set. Moreover, \( Z \) is compact, which is implied by the following facts: a) any sequence \( \{\xi_n\} \) contains a subsequence weakly convergent to an element \( \xi_0(y) \) (because the ground space is separable), and b) the limit of samples is a sample, i.e., \( \xi_0(y) \in Q(y) \). Then it follows from (45) that the limit element \( z_0 = \int_{K_y} \xi_0(y)\sigma(dy) \) belongs to \( Z \). This completes the proof of Lemma 1.

Consider a multivalued mapping \( Q(y,t) \) depending on a parameter \( t \in [t_0,t_1] \). Let us define classes \( \overline{W}_y \) and \( W_y \). We say that \( Q(y,t) \in \overline{W}_y \) provided that

1) \( Q(y,t) \) is a compact convex set for any chosen \( y \) and \( t \),
2) \( \text{gr} Q(y,t)|_{K_y} \) is compact for any \( t \) and \( K_y \),
3) \( \text{gr} Q(\cdot,\cdot) \) is a Borel set, and
4) \( S(t) = \max\{|[\xi(y,t)|_y,\xi(y,t)] \in \text{gr} Q(y,t)|_{K_y}\} \in L_1 \) for each of the sets \( K_y \).

We say that \( Q(y,t) \in W_y \) provided that

1) \( Q(y,t) \) is a compact convex set for any chosen \( y \) and \( t \),
2) \( \text{gr} Q(y,t)|_{K_y} \) is a compact set for any \( t \) and \( K_y \),
3') for each linear functional \( l \in \mathbb{R}^d \), the function
\[ \psi(y,t) = \max_{\xi \in Q} \{l,\xi(y,t)\} \] is a normal integrand, \( \psi(y,t) \) is measurable with respect to \( t \) for chosen \( y \),

where \( f_n(y,t) \) are Carathéodory integrands, and
4) \( S(t) = \max\{|[\xi(y,t)|_y,\xi(y,t)] \in \text{gr} Q(y,t)|_{K_y}\} \in L_1 \) for every \( K_y \).

Let us show that if \( Q(y,t) \in \overline{W}_y \), then \( Q(y,t) \in W_y \).

Lemma 2. Let \( Q(y,t) \in \overline{W}_y \). If \( \varphi(\xi,y) \) is a continuous function of two variables, then the function
\[ \psi(t) = \max_{\xi \in Q} \{\varphi(\xi,y) : y,\xi(y,t) \in \text{gr} Q(y,t)|_{K_y}\} \] is measurable for each \( K_y \).

Proof. We set \( M = \{(\xi,y,t) : \varphi(\xi,y) \geq N\} \). It follows from (47) that \( \psi(t) \geq N \). Therefore, \( (\xi,y,t) \in \text{gr} Q(y,t) \). By condition 3) of the definition of the class \( \overline{W}_y \), the set \( M \) is Borel. By considering the projection of this set to the \( t \) axis we obtain \( \text{Pr}_t M = \{t : \psi(t) \geq N\} \) is a measurable set. This proves Lemma 2.

Lemma 3. Let \( l \) be a linear functional in \( \mathbb{R}^d \) and let \( Q(y,t) \in \overline{W}_y \). Then the function
\[ \psi(y,t) = \max\{(l,\xi(y,t)) : \xi(y,t) \in \text{gr} Q(y,t)\} \] is a normal integrand.

Proof. Let us show that the function \( \psi(y,t) \) satisfies the assumptions of the theorem in [7]. Indeed, by Lemma 2, the function \( \psi(y,t) \) is measurable with respect to \( t \) for chosen \( y \), because the maximum in (47) is taken over \( \text{gr} Q|_{K_y} \) for each of the sets \( K_y \) and, in particular, for the singleton \( y \). This function is upper semicontinuous, because the maximum of
continuous functions over a closed set is an upper semicontinuous function. It remains to verify condition (44). Consider the expression
\[
\max_{y \in K_y} (\psi(y, t) - \varphi(y)),
\]
where \( \varphi(y) \) is a continuous function. However,
\[
\max_{y \in K_y} \psi(y, t) = \max_{y \in K_y} \max_{\xi(y, t) \in Q} (l, \xi(y, t)) = \max_{gr Q | K_y} (l, \xi(y, t)).
\]
Applying Lemma 2, we see that the function (48) is measurable. Then the function \( \psi(y, t) \) is a normal integrand.

Thus, we have shown that if \( Q(y, t) \in \widetilde{W}_y \), then \( Q(y, t) \in W_y \).

In what follows we shall work in the class \( W_y \). We have introduced the class \( \widetilde{W}_y \) in order to show that condition 3') in the definition of \( W_y \) is not a restrictive requirement and readily follows from condition 3) of the definition of \( \widetilde{W}_y \).

We set
\[
Z(t) = \int_{K_y} Q(y, t) \sigma(dy, t),
\]
where \( \sigma(dy, t) \in V_{K_y} \) (see the definition of \( V_{K_y} \) in Part 1).

Lemma 4. Let \( Q(y, t) \in W_y \) and \( \sigma(dy, t) \in V_{K_y} \). Then \( \max_{z \in Z(t)} (l, z(t)) \) is a measurable function for any linear functional \( l \in \mathbb{R}^{d(x)} \).

Proof. We have
\[
\max_{z \in Z(t)} (l, z(t)) = \int_{K_y} \max_{\xi \in Q(y, t)} (l, \xi(y, t)) \sigma(dy, t).
\]
By condition (46), the function \( \max_{\xi \in Q(y, t)} (l, \xi(y, t)) \) is a normal integrand, and hence
\[
\max_{\xi \in Q(y, t)} (l, \xi(y, t)) = \lim_{n \to \infty} \psi_n(y, t),
\]
where \( \psi_n(y, t) \) are Carathéodory integrands. Then it follows from (50) that
\[
\max_{z \in Z(t)} (l, z(t)) = \int \lim_{n \to \infty} \psi_n(y, t) \sigma(dy, t) = \lim_{n \to \infty} \int \psi_n(y, t) \sigma(dy, t).
\]
We must prove that \( \int \psi_n(y, t) \sigma(dy, t) \) is a sequence of measurable functions. This can be proved in two ways, which are based on two known facts:

1) \( \int \varphi(y) \sigma(dy, t) \) is a measurable function for any continuous function \( \varphi(y) \), and
2) \( \int \varphi(y, t) \sigma(dy) \) is a measurable function for any probability measure \( \sigma \) and we have \( \varphi(y, t) \in L_y \) (for the definition of \( L_y \), see Part 1).

In the first case, we must approximate the function \( \psi_n(y, t) \) \( (n = 1, \ldots) \) by a sequence of continuous functions on the full-measure set \( \mathcal{E} \in [t_0, t_1] \). In the second case, we must approximate the measure \( \sigma(dy, t) \) by a sequence \( \sigma_n(dy) \). In both cases we see that \( \int \psi_n(y, t) \sigma(dy, t) \) is a measurable function for any \( n \) \( (n = 1, 2, \ldots) \). Then the assertion of the lemma follows from (52).

We introduce the weak convergence of measures \( (\sigma_n(dy, t) \xrightarrow{\text{weakly}} \sigma(dy, t)) \) in the same way as in Part 1.

**Theorem 1.** Let \( \zeta_n(t), \zeta_0(t) \in L_1[t_0, t_1], \) let \( \zeta_n(t) \xrightarrow{L^\infty \text{weakly}} \zeta_0(t), \) let \( Q(y, t) \in W_y, \) and let \( \sigma_n(dy, t) \xrightarrow{\text{weakly}} \sigma(dy, t). \) If \( \zeta_n(t) \in Z_n(t) = \int_{K_y} Q(y, t) \sigma_n(dy, t), \) then \( \zeta_0(t) \in Z(t) = \int_{K_y} Q(y, t) \sigma(dy, t) \) for any \( t \in [t_0, t_1]. \)
Proof. Let $K_y$ be a common support of the measures $\sigma$ and $\sigma_n$. The following chain of inequalities holds:

$$\int_{t_0}^{t_1} h(t) \min_{\zeta \in \mathcal{Z}(t)} (l, \zeta(t)) \, dt \leq \lim_{n \to \infty} \int_{t_0}^{t_1} h(t) \min_{\zeta_n \in \mathcal{Z}_n(t)} (l, \zeta_n(t)) \, dt$$

$$\leq \int_{t_0}^{t_1} h(t)(l, \zeta_0(t)) \, dt \leq \lim_{n \to \infty} \int_{t_0}^{t_1} h(t) \max_{\zeta \in \mathcal{Z}(t)} (l, \zeta_n(t)) \, dt$$

$$\leq \int_{t_0}^{t_1} h(t) \max_{\zeta \in \mathcal{Z}(t)} (l, \zeta(t)) \, dt \quad \forall t \in \mathbb{R}^d(x); \quad h(t) > 0; \quad h(t) \in L_\infty.$$  

Let us prove the first inequality in this chain. We have

$$\min_{\zeta \in \mathcal{Z}(t)} (l, \zeta(t)) = \int_{K_y} \min_{\xi \in \mathcal{Q}(y,t)} (l, \xi(y,t)) \sigma(dy,t),$$  

$$\min_{\zeta_n \in \mathcal{Z}_n(t)} (l, \zeta_n(t)) = \int_{K_y} \min_{\xi \in \mathcal{Q}(y,t)} (l, \xi(y,t)) \sigma_n(dy,t).$$  

Since $Q(y,t) \in W_y$, it follows that $\min_{\xi \in \mathcal{Q}(y,t)} (l, \xi(y,t))$ is a normal integrand. Therefore,

$$\min_{\xi \in \mathcal{Q}(y,t)} (l, \xi(y,t)) = \sup_k \varphi_k(y,t).$$  

where $\varphi_k(y,t)$ are Carathéodory integrands. It follows from (54), (55), and (56) that

$$\min_{\zeta \in \mathcal{Z}(t)} (l, \zeta(t)) \geq \int_{K_y} \varphi_k(y,t) \sigma(dy,t),$$

$$\min_{\zeta_n \in \mathcal{Z}_n(t)} (l, \zeta_n(t)) \geq \int_{K_y} \varphi_k(y,t) \sigma_n(dy,t).$$  

By assumption we have $\sigma_n \xrightarrow{\text{weakly}} \sigma$, and therefore

$$\int_{K_y} \varphi_k(y,t) \sigma_n(dy,t) \to \int_{K_y} \varphi_k(y,t) \sigma(dy,t).$$

We write $\int_{K_y} \varphi_k(y,t) \sigma_n(dy,t) = b_{kn}$, $\int_{K_y} \varphi_k(y,t) \sigma(dy,t) = a_k$, $\min_{\zeta \in \mathcal{Z}(t)} (l, \zeta(t)) = a$, $\min_{\zeta_n \in \mathcal{Z}_n(t)} (l, \zeta_n(t)) = b_n$, and $\lim_{n \to \infty} \min_{\zeta_n \in \mathcal{Z}_n(t)} (l, \zeta_n(t)) = b$. We must show that $a \leq b$, i.e.,

$$\lim_{k \to \infty} \lim_{n \to \infty} b_{kn} \leq \lim_{n \to \infty} \lim_{k \to \infty} b_{kn}. $$

(59)

This inequality is a consequence of the following well-known inequality that holds for any function $f(x,y)$:

$$\sup_y \inf_x f(x,y) \leq \inf_y \sup_x f(x,y). $$

(60)

In order to pass from (60) to (59), we fix $n$ in the function $b_{kn}$ of integral arguments and write out the following obvious inequality: $\inf_{n > n} b_{kn} \leq b_{kn}$. Furthermore, we obtain the following obvious inequalities: $a_n = \sup_k \inf_{n > n} b_{kn}$ and $a_n = \sup_k \inf_{n > n} b_{kn}$.

\[ \text{sup}_n b_{kn} = \beta_n. \]
On passing to the limit in the last inequality as \( n \to \infty \), we obtain (59). Thus, we have proved that \( \min_{\xi \in Z(t)} (l, \zeta(t)) \leq \lim_{n \to \infty} \min_{\xi \in Z_n(t)} (l, \zeta_n(t)) \). However, this implies the inequality
\[
\int_{t_0}^{t_1} h(t) \min_{\xi \in Z(t)} (l, \zeta(t)) \, dt \leq \lim_{n \to \infty} \int_{t_0}^{t_1} h(t) \min_{\xi \in Z_n(t)} (l, \zeta_n(t)) \, dt,
\]
\( h(t) > 0; \ h(t) \in L_\infty. \) \hfill (61)

Since \( \min_{\xi \in Z(t)} (l, \zeta_n(t)) \leq (l, \zeta_n(t)) \) and by assumption we have \( \zeta_n(t) \xrightarrow{L_\infty-weakly} \zeta_0(t) \), it follows that we can continue inequality (61):
\[
\int_{t_0}^{t_1} h(t) \min_{\xi \in Z(t)} (l, \zeta(t)) \, dt \leq \lim_{n \to \infty} \int_{t_0}^{t_1} h(t) \min_{\xi \in Z_n(t)} (l, \zeta_n(t)) \, dt \leq \lim_{n \to \infty} \int_{t_0}^{t_1} h(t)(l, \zeta_n(t)) \, dt = \int_{t_0}^{t_1} h(t)(l, \zeta_0(t)) \, dt.
\] \hfill (62)

The second half of the chain (53) can be proved similarly.

Thus, we obtain the inequalities
\[
\int_{t_0}^{t_1} h(t) \min_{\xi \in Z(t)} (l, \zeta(t)) \, dt \leq \int_{t_0}^{t_1} h(t)(l, \zeta_0(t)) \, dt \leq \int_{t_0}^{t_1} h(t) \max_{\xi \in Z(t)} (l, \zeta(t)) \, dt.
\] \hfill (63)

It follows from (63) that \( \min_{\xi \in Z(t)} (l, \zeta(t)) \leq (l, \zeta_0(t)) \leq \max_{\xi \in Z(t)} (l, \zeta(t)) \). We can readily prove that there exists a full-measure set on \([t_0, t_1]\) on which, for any \( l \), the following inequality holds:
\[
\min_{\xi \in Z(t)} (l, \zeta(t)) \leq (l, \zeta_0(t)) \leq \max_{\xi \in Z(t)} (l, \zeta(t)). \] \hfill (64)

Indeed, let us take a countable dense set \( \{l_s\} \) of functionals on the unit sphere and set \( \mathcal{E}_s = \{ t : \min_{\xi \in Z(t)} (l_s, \zeta(t)) \leq (l_s, \zeta_0(t)) \leq \max_{\xi \in Z(t)} (l_s, \zeta(t)) \} \). By Lemma 4, \( \mathcal{E}_s \) is a measurable set of full measure. Then the set \( \mathcal{E} = \bigcap \mathcal{E}_s \) is also a measurable set of full measure on \([t_0, t_1]\). On the set \( \mathcal{E} \), inequality (64) holds for each of the functionals \( l_s \), and hence for any \( l \) (because \( Z(t) \) is compact). It follows from (64) that \( \zeta_0(t) \in Z(t) \) for any \( t \in \mathcal{E} \), i.e., \( \zeta_0(t) \in Z(t) \). This completes the proof of Theorem 1.

§2. STATEMENT AND PROOF OF THE MAIN RESULTS

On the basis of the results of the preceding section we can describe the entire class of limit inclusions for (33) as \( u_n(t) \xrightarrow{weakly} 0 \). Let us write out relation (33) in the following equivalent form:
\[
S_n : \dot{z}(\varepsilon) \int_{K_u} \tilde{f}(x, u, t) \mu_n(du, t),
\] \hfill (65)

where \( \mu_n(du, t) = E_{u_n}(du, t) = \delta(u - u_n(t)) \, du \). Assume that \( \mu_n \xrightarrow{weakly} \mu \). Consider the inclusion
\[
S : \dot{z}(\varepsilon) \int_{K_u} \tilde{f}(x, u, t) \mu(du, t).
\] \hfill (66)

We say that \( z_n \in S_n \) if \( z_n(t) \) satisfies relation (65) and that \( z \in S \) if \( z(t) \) satisfies relation (66).

Denote a pair \( (x, u) \) by \( y \), i.e., \( y \in K_x \times K_u = K_y \). In what follows, the notation \( K_{xy} \) and \( W_{xy} \) will correspond to \( K_y \) and \( W_y \), respectively.

Let us prove the following assertion.
Theorem 2. Let \( f(x, u, t) \in W_{xu} \), let
\[
\mu_n(du, t) \overset{\text{weakly}}{\longrightarrow} \mu(du, t),
\]
and let \( x_n(t) \in S_n \). Then
\[
x_n(t) \subseteq x^0(t) (\in S).
\]

Proof. Let \( x_n(t) \in S_n \) and \( x^0(t) (\in S) \). Then we have
\[
\int_{K_u} f(x_n, u, t) \mu_n(du, t) = \int_{K_u} f(x, u, t) \mu_n(du, t) E_{x_n}(dx, t) = Z_n(t),
\]
\[
\int_{K_u} f(x^0, u, t) \mu(du, t) = \int_{K_u} f(x, u, t) \mu(du, t) E_{x^0}(dx, t) = Z(t).
\]
We write \( \mu_n(du, t) \times E_{x_n}(dx, t) = \sigma_n(du, dx, t) \) and
\[
\mu(du, t) \times E_{x^0}(dx, t) = \sigma(du, dx, t).
\]
By assumption, \( \mu_n(du, t) \overset{\text{weakly}}{\rightarrow} \mu(du, t) \) and \( E_{x_n}(dx, t) \overset{\text{weakly}}{\rightarrow} E_{x^0} \), and this implies the relation
\[
\sigma_n(du, dx, t) \overset{\text{weakly}}{\rightarrow} \sigma(du, dx, t).
\]
Since \( x_n(t) \subseteq x^0(t) \), it follows that \( \dot{x}_n(t) \overset{L^\infty-\text{weakly}}{\rightarrow} \dot{x^0}(t) \). Thus, the conditions of Theorem 2 are reduced to those of Theorem 1, and Theorem 1 was proved in the preceding section.

Theorem 2 allows us to describe the entire class of limit inclusions of the form (33) in dependence on the parameter \( \mu(du, t) \), namely,
\[
\dot{x} (\in) \int_{K_u} f(x, u, t) \mu(du, t).
\]
As a rule, the limit inclusion is "good," i.e., it is not an inclusion (it is not multivalued) but an ordinary equation. In the general case, the theorem converse to Theorem 2 fails. The following theorem holds.

Theorem 3. Assume that \( f(x, u, t) \in W_{xu} \) and that \( x^0(t) \) is a unique solution of the inclusion \( (66) \). Let there exist a sequence of solutions \( \{x_n(t)\} \in S_n \) (65) with fixed initial data \( \xi_n(t_0) \), and let the sequence \( \{x_n(t)\} \) be bounded. If \( \xi_n(t_0) \rightarrow x^0(t_0) \) and \( \mu_n \overset{\text{weakly}}{\rightarrow} \mu \), then \( x_n(t) \rightarrow x^0(t) \) with respect to the norm of the space \( \mathbb{C} \).

Proof. The systems \( S_n \) and \( S \) are equivalent to the following systems of integral equations:
\[
x_n(t) \in \xi_n(t_0) + \int_{t_0}^{t} \int_{K_u} f(x_n, u, \tau) \mu_n(du, \tau), \tag{68}
\]
\[
x^0(t) \in x^0(t_0) + \int_{t_0}^{t} \int_{K_u} f(x^0, u, \tau) \mu(du, \tau), \tag{69}
\]
where \( \mu_n(du, t) = E_{u_n}(du, t) \). Since \( f(x, u, t) \in W_{xu} \) and the family \( \{x_n(t)\} \) is bounded, we can readily show that this family satisfies the conditions of the Arzela theorem, and thus \( x_n \subseteq y^0 \). Since, by assumption, the right-hand side of (68) converges to the right-hand side of (69), it follows from the uniqueness of solution to the inclusion (69) that \( y^0(t) = x^0(t) \). This completes the proof of Theorem 3.

§3. EXAMPLES

Let us illustrate the above results by known problems in which vibration acts on mechanical systems with dry friction.
Consider the one-dimensional case.

Let a mass point \((m = 1)\) be subjected to the action of a resisting force \(F(x)\) (we assume that this is the force of dry friction) and belong to a coarse plane performing longitudinal small oscillations according to the law \(a(t)\) with \(a(t) \overset{\text{weakly}}{\rightarrow} 0\) and \(\dot{a}(t) \overset{\text{weakly}}{\rightarrow} 0\). The motion relations can be written as follows:

\[
\ddot{x} = F(\dot{x}) + \varphi(\ddot{a}, \dot{x}),
\]

where

\[
F(\dot{x}) = \begin{cases} 
1, & \dot{x} < 0, \\
0, & \dot{x} = 0, \\
-1, & \dot{x} > 0
\end{cases}
\]

(here we take \(fN = 1\), where \(f\) is the coefficient of sliding friction and \(N\) is the reaction of the rest), and

\[
\varphi(\ddot{a}, \dot{x}) = \begin{cases} 
\ddot{a}, & \dot{x} < 0, \\
\ddot{a}, & \dot{x} > 0, \\
0, & \dot{x} = 0, \\
\ddot{a}, & \dot{x} = 0
\end{cases}
\]

where \(f_1\) is the force of static friction (as a rule, \(f_1 > fN\)). We make the change of variables

\[
x - a = y.
\]

Then Eq. (70) can be rewritten as follows:

\[
\ddot{y} = f(\dot{y} + \dot{a}) = \begin{cases} 
1, & \dot{y} + \dot{a} < 0, \\
-1, & \dot{y} + \dot{a} > 0, \\
\ddot{a}, & \dot{y} + \dot{a} = 0, \\
0, & \dot{y} + \dot{a} = 0
\end{cases}
\]

It follows from (74) that, for \(\dot{y} + \dot{a} = 0\), the right-hand side has a value belonging to the segment \([-f_1, f_1]\) if \(f_1 > 1\). Taking the convex closure of the right-hand side we obtain the differential inclusion

\[
\ddot{y} \in \tilde{f}(\dot{y} + \dot{a}),
\]

where

\[
\tilde{f}(\dot{y} + \dot{a}) = \begin{cases} 
1, & \dot{y} + \dot{a} < 0, \\
-1, & \dot{y} + \dot{a} > 0, \\
[-f_1, f_1], & \dot{y} + \dot{a} = 0.
\end{cases}
\]

As follows from the preceding section, the limit differential inclusion has the form

\[
y \in \int_{\mathcal{K}_u} \tilde{f}(\dot{y} + u)\mu(du, t),
\]

where \(\mu(du, t)\) is the limit measure for the measures \(E_{\dot{a}}(du, t)\), and \(\mathcal{K}_u\) is a support of the measure.

In dependence on the form of the function \(\dot{a}(t)\) we shall find the limit measure and obtain limit differential inclusions (or equations) with various properties. If the plane oscillates...
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along the $x$ axis according to the law $\dot{a}(t) = A \sin \omega t$, we can readily obtain (as is known from [8]) the effect of seeming conversion of dry friction into viscous friction; however, there is no vibrodisplacement (without external force). Indeed, in this case the limit measure does not depend on $t$, is concentrated on the segment $[-A, A]$, and has the form

$$\mu(du, t) = \frac{du}{A \pi \sqrt{1 - (u/A)^2}}. \quad (78)$$

Let us substitute this measure into (77) and take account of (76). Then we obtain the following function on the right-hand side:

$$\Phi(\dot{y}) = \frac{1}{A \pi} \int_{-A}^{A} \tilde{f}(\dot{y} + u) \frac{du}{\sqrt{1 - (u/A)^2}} = \frac{1}{A \pi} \int_{-A}^{\dot{y} + A} \tilde{f}(z) \frac{dz}{\sqrt{1 - ((z - \dot{y})/A)^2}}, \quad (79)$$

or, in other words,

$$\Phi(\dot{y}) = \begin{cases} 
1, & \dot{y} \leq -A, \\
-1, & \dot{y} \geq A, \\
-(2/\pi) \arcsin(\dot{y}/A), & -A \leq \dot{y} \leq A. 
\end{cases} \quad (80)$$

Thus, on the right-hand side of (77), we have a single-valued continuous function satisfying the Lipschitz condition, i.e., instead of a differential inclusion, we have obtained an ordinary equation, which has a unique solution. Since it follows from (73) that $y \to x$ as $a \to 0$, it follows that this equation has the form $\ddot{x} = \Phi(\dot{x})$. For $\dot{x} = 0$ we have $\Phi(0) = 0$, and hence we have no vibrodisplacement.

Consider the biharmonic law of the vibrational action

$$\dot{a}(t) = \cos \omega t + \cos 2\omega t. \quad (81)$$

As in known [8], in this case we can see the motion without external force.

The limit measure also does not depend on $t$; it is the sum of measures $\mu_1(du) + \mu_2(du)$, where

$$\mu_1(du) = \frac{4}{\pi} \frac{du}{\sqrt{6 - 8u + 2\sqrt{9 + 8u} \sqrt{9 + 8u}}}, \quad (82)$$

$$\mu_2(du) = \frac{4}{\pi} \frac{du}{\sqrt{6 - 8u - 2\sqrt{9 + 8u} \sqrt{9 + 8u}}}. \quad (83)$$

The measure $\mu_1$ is concentrated on the segment $[-9/8, 2]$ and the measure $\mu_2$ on the segment $[-9/8, 0]$. After integrating the function $\tilde{f}(\dot{y} + u)$ (76) with respect to the limit measure we obtain a single-valued smooth function on the right-hand side of (77):

$$\Phi(\dot{y}) = \begin{cases} 
1, & \dot{y} \leq -2, \\
-1, & \dot{y} \geq 9/8, \\
\int_{-9/8}^{2} \tilde{f}(\dot{y} + u) \mu_1(du) + \int_{-9/8}^{0} \tilde{f}(\dot{y} + u) \mu_2(du), & -2 \leq \dot{y} \leq 9/8. 
\end{cases} \quad (84)$$

Taking account of (76) we can write the values $\Phi(\dot{y})$ for $\dot{y} \in [-2, 9/8]$ in the form

$$\int_{-9/8}^{\dot{y}} \mu_1(du) - \int_{\dot{y}}^{2} \mu_1(du) + \int_{-9/8}^{0} \mu_2(du), \quad \dot{y} \in [-2, 0],$$

$$\int_{-9/8}^{\dot{y}} \mu_1(du) - \int_{\dot{y}}^{2} \mu_1(du) + \int_{-9/8}^{\dot{y}} \mu_2(du) - \int_{\dot{y}}^{0} \mu_2(du), \quad \dot{y} \in [0, 9/8].$$
If we successively make the change of variables

$$z = \frac{\sqrt{9+8u} + 8u}{(z+3)}$$

in the integrals with respect to the measure $\mu_1(du)$ and

$$z = \frac{\sqrt{9+8u} + 8u}{(z+5)}$$

in the integrals with respect to $\mu_2(du)$, we finally obtain $\Phi(\dot{y})$ on the segment $[-2,9/8]$:  

$$\Phi(\dot{y}) = \begin{cases} 
1, & \dot{y} \leq -2, \\
-1, & \dot{y} \geq 9/8, \\
\frac{4}{\pi} \arctg \sqrt{\frac{\sqrt{9-8y} + 3}{\sqrt{9-8y} - 5} - 1}, & \dot{y} \in [-2,0], \\
\frac{4}{\pi} \arctg \sqrt{\frac{\sqrt{9-8y} - 3}{\sqrt{9-8y} + 5} - 1}, & \dot{y} \in [0,9/8].
\end{cases} \tag{85}$$

Taking account of (73) in the passage to the limit as $\alpha \to 0$ we obtain the equation

$$\ddot{x} = \Phi(\dot{x}), \tag{86}$$

where $\Phi(x)$ is the function (85), which is single-valued, continuous, and satisfying the Lipschitz condition. Since $\Phi(0) = 1/3$, it follows that under the vibration (81) the motion starts without external force.

To find out if the motion starts under the action of vibration, we do not need to calculate a value of the limit function on the right-hand side of (86) at $\dot{x} = 0$. It suffices to analyze the graph of $\dot{x}(t)$. If the sum of lengths of the intervals on which $\dot{x}(t) > 0$ is not equal to the sum of lengths of the intervals on which $\dot{x}(t) < 0$, then the point starts the motion from the point of rest under the action of such a vibration without external force. Otherwise the body remains at rest. This can readily be shown on the basis of the definition of weak convergence of measures (see Part I) for $f = 1$. This condition is sufficient, in contrast to the "nonsymmetry" condition for the law of vibrational action [9] which is only necessary.

II. Two-dimensional case. A body ($m = 1$) not subjected to the action of an external force is in relative rest on a coarse vibrating plane. The projections of the dry friction force are defined by the relations

$$F_x(x,y) = -\dot{x}/\sqrt{\dot{x}^2 + \dot{y}^2}, \tag{87}$$

$$F_y(x,y) = -\dot{y}/\sqrt{\dot{x}^2 + \dot{y}^2} \tag{88}$$

(here we again set $fN = 1$, where $f$ is the coefficient of sliding friction and $N$ is the reaction of the rest). Thus, at any point of the plane, a unit field is given. At the point (0,0) this field is undefined, and we set $F_x(0,0) = F_y(0,0) = 0$. Denote the vector $(\dot{x}, \dot{y})$ by $\dot{w}$. Then the motion relation in vector form, with vibration taken into account, can be written as follows:

$$\ddot{w} = \bar{F}(\dot{w} + \varphi(\dot{w}, \ddot{a})), \tag{89}$$

where

$$\bar{F}(\dot{w}) = \begin{cases} 
(F_x(\dot{x}, \dot{y}), F_y(\dot{x}, \dot{y})), & \dot{w} \neq 0, \\
(0,0), & \dot{w} = 0.
\end{cases}$$

$$\varphi(\dot{w}, \ddot{a}) = \begin{cases} 
0, & \dot{w} = 0, \quad |\ddot{a}| \leq f_1, \\
\ddot{a}, & \dot{w} = 0, \quad |\ddot{a}| \geq f_1.
\end{cases}$$
and \( f_1 \) stands for the force of static friction. On making the change of variables

\[
\mathbf{w} - \mathbf{a} = \mathbf{w}_1, \quad (\mathbf{w}_1 = (x_1, y_1))
\]

and arguing similarly to the one-dimensional case, we obtain the following convex differential inclusion:

\[
\mathbf{w}_1 \in \mathbf{F}(\mathbf{w} + \mathbf{a}), \quad \mathbf{F}(\mathbf{w} + \mathbf{a}) = \begin{cases} 
(\mathbf{F}_x(\mathbf{w}_1 + \mathbf{a}), \mathbf{F}_y(\mathbf{w}_1 + \mathbf{a})), & \mathbf{w}_1 + \mathbf{a} \neq \mathbf{0}, \\
\mathcal{B}_{f_1}, & \mathbf{w}_1 + \mathbf{a} = \mathbf{0},
\end{cases}
\]

where \( \mathcal{B}_{f_1} \) is a circle of radius \( f_1 \) with center at the point \((0,0)\). For the direction of vibration, we choose the one along the \( y \) axis, i.e., we set \( \mathbf{a} = (0, a_2) \). Then we have

\[
\mathbf{F}_x = -\frac{x}{\sqrt{x^2 + (y_1 + a_2)^2}}, \quad \mathbf{F}_y = -\frac{y_1 + a_2}{\sqrt{x^2 + (y_1 + a_2)^2}}.
\]

Assume that the vibration is such that the limit measure is the Lebesgue measure on the segment \([-1/2,1/2]\). Let us integrate the vector \( \mathbf{F} \) (91) with respect to this measure. Then we obtain the limit value of this vector with the components

\[
\mathbf{F}_1(x, y_1) = \int_{-1/2}^{1/2} \mathbf{F}_x(x, y_1 + u) \, du, \quad \mathbf{F}_2(x, y_1) = \int_{-1/2}^{1/2} \mathbf{F}_y(x, y_1 + u) \, du.
\]

With regard for (90) we obtain by passing to the limit as \( \alpha \to 0 \) that \( y_1 = y \). Then we have

\[
\mathbf{F}_1(x, y) = -x \ln \frac{\sqrt{x^2 + (y + 1/2)^2} + y + 1/2}{\sqrt{x^2 + (y - 1/2)^2} + y - 1/2},
\]

\[
\mathbf{F}_2(x, y) = -\sqrt{x^2 + (y + 1/2)^2} + \sqrt{x^2 + (y - 1/2)^2}.
\]

The functions \( \mathbf{F}_1(x, y) \) and \( \mathbf{F}_2(x, y) \) are single-valued continuous functions of their arguments, i.e., the limit relation is a differential equation (and not an inclusion). Let us show that the system

\[
x = \mathbf{F}_1(x, y), \quad y = \mathbf{F}_2(x, y)
\]

has a unique solution. The function \( \mathbf{F}_2(x, y) \) has continuous derivatives with respect to both arguments. The function \( \mathbf{F}_1(x, y) \) has an infinite derivative with respect to \( x \) at the point \((0,0)\), i.e., the system (92) does not satisfy the classical uniqueness theorem. Nevertheless the system has a unique solution. Indeed, let us present the function \( \mathbf{F}_1(x, y) \) in the form

\[
\mathbf{F}_1(x, y) = 2x \ln |x| + \mathbf{F}_1(x, y),
\]

where \( \mathbf{F}_1(x, y) \) is a smooth function given by the relation

\[
\mathbf{F}_1(x, y) = -x \ln \left( \frac{(\sqrt{x^2 + (y + 1/2)^2} + y + 1/2)}{(\sqrt{x^2 + (y - 1/2)^2} + y - 1/2)} \right).
\]

Since the function \( x \ln |x| \) enters the origin at infinite time, the system (92) has a unique solution.

Thus, we have shown that the dry friction converts into the viscous friction.

This example models the problem of downward rolling logs [10]. If we incline the plane a little and direct the \( x \) axis along the line of maximal slope and the \( y \) axis perpendicularly, then under the described vibration the logs will roll down under the action of the force of gravity.
REFERENCES


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