Conditions for the non-negativity of integral quadratic forms with constant coefficients on a half-axis

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Abstract. An integral quadratic functional with constant coefficients on a half-axis is considered. A necessary and sufficient condition for its non-negativity at all square integrable pairs of functions related by a linear ODE is proposed, which is based on the Hamilton-Jacobi inequality. A connection between this condition and the well-known frequency criterion is established.

Bibliography: 3 titles.

§ 1. Statement of the problem

Let \( x \in \mathbb{R}^{d(x)}, u \in \mathbb{R}^{d(u)}, \) and \( d(u) \leq d(x) \) (\( d(a) \) denotes the dimension of a vector \( a \)). Denote by \( L \) the linear variety of all finite pairs \( (x(t), u(t)) \mid t \in [0, \infty) \), that is, pairs that are distinct from zero only on a bounded set. The function \( x(\cdot) \) is assumed to be absolutely continuous and equal to zero at \( t = 0 \), and the function \( u(t) \) is assumed to be measurable and essentially bounded on \([0, \infty)\).

Consider a control system \( V: \dot{x} = Kx + Bu \), where \( K \) and \( B \) are constant matrices. Denote by \( L(V) \) the set of all elements in \( L \) that are trajectories of the system \( V \). We set

\[
\Phi(x, u) = u^2 + 2Axu + Qxx,
\]

where

\[
A: \mathbb{R}^{d(x)} \rightarrow \mathbb{R}^{d(u)}, \quad Q: \mathbb{R}^{d(x)} \rightarrow \mathbb{R}^{d(x)}
\]

are constant matrices and \( Q \) is assumed to be symmetric.

For a pair \((x(\cdot), u(\cdot)) \in L\) we define the functional

\[
J(x(\cdot), u(\cdot)) = \int_{0}^{\infty} \Phi(x(t), u(t)) \, dt.
\]

We are interested in necessary and sufficient conditions for the inequality

\[
J(x, u) \geq 0 \quad \text{for all } (x, u) \in L(V).
\]

The condition that we obtain is formally weaker, and therefore more advanced, than the well-known condition derived from the Fourier expansion [1].

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§ 2. Formulation of the main theorem

Let \( A, Q, K, B \) be fixed matrices. We shall denote by \( S \) an arbitrary symmetric matrix. We fix such a matrix \( S \). For a trajectory of the system \( V \) we set

\[
\Phi_S(x(\cdot), u(\cdot)) = \Phi(x(\cdot), u(\cdot)) + \frac{d}{dt} (S x(t), x(t)).
\]

One can easily see that

\[
\Phi_S(x, u) = (u + U_S x)^2 + \omega_S(x),
\]

where

\[
U_S : \mathbb{R}^d(x) \rightarrow \mathbb{R}^d(u), \quad U_S = -(A + B^* S),
\]

\[
\omega_S(x) = -(U_S x)^2 + 2(K x, S x) + (Q x, x).
\]

For \((x(\cdot), u(\cdot)) \in L(V)\) let

\[
J_S(x(\cdot), u(\cdot)) = \int_0^\infty \Phi_S(x(t), u(t)) \, dt.
\]

Obviously,

\[
J_S(x(\cdot), u(\cdot)) = J(x(\cdot), u(\cdot)) \quad \forall (x, u) \in L(V).
\]

Let \( \lambda(S) \) be the minimum eigenvalue of the form \( \omega_S(x) \). We set

\[
\hat{\lambda} = \sup_S \lambda(S).
\]

The main result of this paper is as follows.

**Theorem 1.** The inequality \( \hat{\lambda} \geq 0 \) is equivalent to the inequality \( J \geq 0 \) on \( L(V) \).

The expression for \( \omega_S(x) \) is the left-hand side of the Hamilton–Jacobi equation for a function of the form \( (S x, x) \) with constant \( S \).

The condition \( \omega_S(x) \geq 0 \) is thus the Hamilton–Jacobi inequality. The sufficiency part of Theorem 1 is obvious. Its main assertion is the necessity. We shall establish it without the use of the frequency criterion [1].

In the case of a two-dimensional space and the control system \( \dot{x} = u \), Dmitruk [2] studied the non-negativity of \( J \) on \( L(V) \) by essentially using Theorem 1, although he did not state it. He obtained a new condition, which is equivalent to the non-negativity of \( J \) on \( L(V) \) and is not connected with the frequency criterion. We now present its formulation.

Without loss of generality we may assume the matrix \( A \) to be skew-symmetric. Let \( q_0 \) and \( q_1 \) be the eigenvalues of the matrix \( Q \), and \( a \) the modulus of the eigenvalue of \( A \). Then the above-mentioned condition consists in the inequalities

\[
q_0 \geq 0, \quad q_1 \geq 0, \quad \sqrt{q_0} + \sqrt{q_1} \geq 2a.
\]

In the next section we present lemmas required for the proof of Theorem 1.
§ 3. Several lemmas

3.1. Let $R : \mathbb{R}^d(x) \to \mathbb{R}^d(x)$ be a linear operator.

Lemma 1. The following two conditions are equivalent:

(*) $\max_{|x|=1} (Rx, Sx) \geq 0$ for each symmetric matrix $S$;

(**) the matrix $R$ has an eigenvalue with real part zero.

Proof. We prove first that

\begin{equation}
(**) \Rightarrow (*) .
\end{equation}

Indeed, if $\lambda = 0$ is an eigenvalue of the matrix $R$, then there exists a vector $x_*$ such that $Rx_* = 0$ and $|x_*| = 1$. Then for each matrix $S$ we have $(Rx_*, Sx_*) = 0$ and (*) holds.

Now let $\lambda = i\sigma$, $\sigma \neq 0$, be an eigenvalue of the matrix $R$. Then it is known that the equation $\dot{x} = Rx$ has a solution of the following form:

$$x^0(t) = x_0 \cos \sigma t + x_1 \sin \sigma t$$

with linearly independent vectors $x_0$ and $x_1$.

Let $S$ be an arbitrary symmetric matrix. Consider the function $\frac{1}{2} S x^0(t)x^0(t)$ and let $t_*$ be a maximum point of it. Then its derivative at $t_*$ vanishes, that is, $(Rx^0(t_*), Sx^0(t_*)) = 0$. However, $x^0(t_*) \neq 0$ since $x_0$ and $x_1$ are linearly independent. Hence $R$ satisfies (*).

We have thus proved (3.1). We prove next that if condition (**) fails to hold, then conditions (*) fail too.

Indeed, it is clear from the Jordan form of the matrix $R$ that the space $\mathbb{R}^d(x)$ is decomposed into a direct sum of $R$-invariant subspaces such that on each of these subspaces the operator $R$ can be represented in one of the following forms.

1. Let $X$ be one of the subspaces. Then, there exists a basis in $X$ such that the operator $R$ has the following form with respect to it:

\begin{align*}
y_1 &= \lambda x_1 + x_2, \\
y_k &= \lambda x_k + x_{k+1}, & k &\leq n - 1, \\
y_n &= \lambda x_n,
\end{align*}

where $\lambda$ is a real number and $n = d(X)$.

2. Let $d(X) = 2m$. We represent elements of $X$ as $(\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_m, \eta_m)$, that is, we denote odd components by $\xi_k$ and even components by $\eta_k$, $k = 1, \ldots, m$. There exists a basis in $X$ in which the operator $R$ has the following form:

\begin{align*}
\bar{w}_1 &= \lambda w_1 + \sigma \bar{V} w_1 + w_2, \\
\bar{w}_k &= \lambda w_k + \sigma \bar{V} w_k + w_{k+1}, & k &\leq m - 1, \\
\bar{w}_m &= \lambda w_m + \sigma \bar{V} w_m,
\end{align*}

where

\begin{align*}
w_k &= (\xi_k, \eta_k), \\
\bar{V}w &= (\bar{\xi}, \bar{\eta}), \\
\bar{\xi} &= -\eta, \\
\bar{\eta} &= \xi, \\
\sigma &\neq 0.
\end{align*}
In the first case $\lambda$ is a real eigenvalue of the operator $R$; in the second $\lambda \pm i\sigma$ is a pair of conjugate eigenvalues of $R$. Hence $\lambda \neq 0$ in either case. We shall speak of invariant subspaces of the first and the second type, depending on whether the corresponding eigenvalue is real or not. In each subspace we define a quadratic form as follows.

Let $X$ be a subspace of the first type. We consider a basis in which $R$ has the form (3.2) and set

$$Z_X(x) = \sum_{k=1}^{d(X)} s_k x_k^2.$$  

We choose the coefficients $s_k$ as follows. We have

$$\frac{1}{2} \frac{d}{dt} Z_X(x) = \sum_{k=1}^{d(X)} \lambda s_k x_k^2 + \sum_{k=1}^{d(X)-1} s_k x_k x_{k+1},$$

where $\frac{dx}{dt} = Rx$. We set sign $s_k = -\text{sign} \lambda$ for each $k$. Moreover, we require $|s_k|$ to increase sufficiently rapidly with $k$, so that for $x \neq 0$ we have

$$\frac{1}{2} \frac{d}{dt} Z_X(x) < 0.$$  

This is obviously feasible.

For a subspace of the second type we set

$$Z_X(w) = \sum_{k=1}^{\frac{1}{2}d(X)} s_k w_k^2,$$

where $w_k^2 = \xi_k^2 + \eta_k^2$.

Then

$$\frac{1}{2} \frac{d}{dt} Z_X(w) = \sum_{k=1}^{\frac{1}{2}d(X)} \lambda s_k w_k^2 + \sum_{k=1}^{\frac{1}{2}d(X)-1} s_k w_k w_{k+1}, \quad \frac{dw}{dt} = Rw.$$  

As before, we set sign $s_k = -\text{sign} \lambda$ and require that the growth of $|s_k|$ guarantees the inequality

$$\frac{1}{2} \frac{d}{dt} Z_X(w) < 0 \quad \text{for } w \neq 0.$$  

Again, this requirement is feasible.

Thus, the forms $Z_X$ are defined. We set $\tilde{Z}(x) = \sum X Z_X(x)$. In the initial basis we have $\tilde{Z}(x) = (\tilde{S}x, x)$, where $\tilde{S}$ is a symmetric matrix. By the construction $\frac{1}{2} \frac{d}{dt} \tilde{Z}(x) < 0$ for all $x \neq 0$. Therefore, $(Rx, \tilde{S}x) < 0$ for $x \neq 0$, so that (*) fails to hold. The lemma is proved.

3.2. Let $\Gamma$ be a non-trivial subspace of $\mathbb{R}^d(x)$. Consider the condition

$$(*)_{\Gamma} \max\{(Rx, Sx) : x \in \Gamma, \ |x| = 1\} \geq 0 \text{ for each symmetric matrix } S.$$
**Lemma 2.** Assume that the condition $(*; \Gamma)$ holds. Then there exists a subspace $\hat{\Gamma}$ such that

$$\hat{\Gamma} \subset \Gamma, \quad d(\hat{\Gamma}) > 0, \quad R\hat{\Gamma} \subset \hat{\Gamma},$$

and the condition $(*; \hat{\Gamma})$ is fulfilled.

**Proof.** For an operator $M: \mathbb{R}^{d(x)} \to \mathbb{R}^{d(x)}$ we define the operators

$$M_1: \Gamma \to \mathbb{R}^{d(x)}, \quad M_1 x = P_T M x,$$

$$M_2: \Gamma \to \mathbb{R}^{d(x)}, \quad M_2 x = P_{\Gamma'} M x,$$

where $P_T$ and $P_{\Gamma'}$ are the orthogonal projections onto $\Gamma$ and onto its orthogonal complement $\Gamma'$, respectively.

Note that the operator $M$ coincides on $\Gamma$ with some symmetric operator $S$ if and only if $M_1$ is a symmetric operator on $\Gamma$. As regards the operator $M_2$, it can be arbitrary. This is easy to prove.

We can now proceed to the proof of the lemma.

If $R_2 = 0$, then the assertion of the lemma holds since $\hat{\Gamma} = \Gamma$ in this case. Assume that $R_2 \neq 0$. Then we claim that $d(R_2 \Gamma) < d(\Gamma)$. Indeed, otherwise we choose $S$ such that $S x = x$, $S^2 = -R_2$, and obtain

$$(S x, R x)|_\Gamma = -(R_2 x)^2.$$ 

However, $d(R_2 \Gamma) = d(\Gamma)$, and therefore $(S x, R x) < 0$ for all $x \in \Gamma, x \neq 0$, which contradicts the condition of the lemma.

Thus, $d(R_2 \Gamma) < d(\Gamma)$. We set $\Gamma_1 = R_2^{-1}(0)$. Obviously, $0 < d(\Gamma_1) < d(\Gamma)$. We now claim that $R$ satisfies the condition $(*; \Gamma_1)$.

Assume that, on the contrary, there exists a symmetric matrix $S$ such that $(S x, R x) < 0$ for all $x \in \Gamma_1, x \neq 0$. We choose $\hat{S}$ such that $\hat{S}_1 = S_1, \hat{S}_2 = -N R_2$, where $N$ is a real coefficient. Then $(\hat{S} x, R x)|_\Gamma = (S_1 x, R_1 x) - N(R_2 x)^2$. We set

$$x = x' + x'', \quad x' \in \Gamma_1, \quad x'', \in \Gamma_1, \quad x, x'' \in \Gamma.$$ 

Then

$$(\hat{S} x, R x)|_\Gamma = (S_1 x', R_1 x') + (S_1 x', R_1 x'') + (S_1 x'', R_1 x') + (S_1 x'', R_1 x'') - N(R_2 x)^2.$$ 

By the definition of $\Gamma_1$ we obtain $R_2 x' = 0$. Hence the last term can be written in the form $-N(R_2 x'')^2$. For the same reason $(S_1 x', R_1 x') = (S x', R x')$, and therefore this quantity has the estimate $-\alpha_0 x'^2$, where $\alpha_0 > 0$. Further, $(R_2 x'')^2 \geq \alpha_1 x''^2$, where $\alpha_1 > 0$, and therefore for sufficiently large $N$ we obtain

$$(\hat{S} x, R x) < 0 \quad \text{for all } x \in \Gamma, \quad x \neq 0,$$

which contradicts the condition $(*; \Gamma)$. We have thus proved the following result: if $\Gamma$ is not invariant with respect to $R$, then there exists $\Gamma_1$ such that $0 < d(\Gamma_1) < d(\Gamma)$ and the condition $(*; \Gamma_1)$ holds. This easily yields the assertion of the lemma.
3.3. Let \( \omega(x, y) = Cxx + Dxy + Eyy \) be a quadratic form on \( \mathbb{R}^{2d(x)} \). We assume that \( E \) is a non-negative-definite matrix. Let \( S \) be a symmetric matrix, and \( \Gamma \) a subspace of \( \mathbb{R}^{d(x)} \). Then we set

\[
f(S, \Gamma) = \max_{x \in \Sigma(\Gamma)} \omega(x, Sx),
\]

where

\[
\Sigma(\Gamma) = \{x \in \Gamma : |x| = 1\}
\]

is the unit sphere in \( \Gamma \).

If \( \Gamma = \{0\} \), then by agreement, \( f(S, \Gamma) = -\infty \) for each \( S \).

We set \( f(\Gamma) = \inf_S f(S, \Gamma) \), and let \( \mathcal{J} \) be the space of symmetric matrices. The scalar product in \( \mathcal{J} \) is defined as the sum of the products of entries with equal indices.

We set

\[
f_0(\Gamma) = \{S : (Dx, Sx) = 0, (ESx, Sx) = 0 \text{ on } \Gamma\}.
\]

Since \( E \geq 0 \), the second condition is equivalent to the relation \( ES = 0 \) on \( \Gamma \). Therefore, \( f_0(\Gamma) \) is a subspace of \( \mathcal{J} \) for each \( \Gamma \). We denote by \( f_1(\Gamma) \) the orthogonal complement to \( f_0(\Gamma) \) in \( \mathcal{J} \).

We consider now the following two sets:

\[
\Omega(\Gamma) = \{S \in f_1(\Gamma) : (Dx, Sx) \leq 0, (ESx, Sx) = 0 \text{ on } \Gamma\},
\]

\[
\Gamma(\Gamma) = \{x \in \Gamma : (Dx, Sx) = 0 \text{ for each } S \in \Omega(\Gamma)\}.
\]

Obviously, \( \Omega(\Gamma) \) is a convex cone in \( f_1(\Gamma) \). In view of the implication

\[
S \in \Omega(\Gamma) \Rightarrow (Dx, Sx) \leq 0 \text{ on } \Gamma,
\]

the condition \( (Dx, Sx) = 0 \) defines a subspace of \( \Gamma \). Hence \( \Gamma(\Gamma) \) is a subspace of \( \Gamma \).

Consider the following sequence generated by the subspace \( \Gamma \). We set

\[
\Gamma_1 = \Gamma(\Gamma), \quad \Gamma_2 = \Gamma(\Gamma_1), \quad \ldots, \quad \Gamma_{k+1} = \Gamma(\Gamma_k), \quad \ldots.
\]

Since \( \Gamma_{k+1} \subset \Gamma_k \), it follows that, beginning from some \( k = k_0 \),

\[
\Gamma_{k_0+1} = \Gamma_{k_0}, \quad \text{so that } \Gamma_k = \Gamma_{k_0} \text{ for each } k \geq k_0.
\]

The subspace \( \Gamma_{k_0} \) will be denoted by \( \tilde{\Gamma}(\Gamma) \). If \( \Gamma = \mathbb{R}^{d(x)} \), then we simply write \( \tilde{\Gamma} \), dropping the argument \( \Gamma \).

**Lemma 3.** The following conditions hold for each \( \Gamma \):

1. \( f(\tilde{\Gamma}(\Gamma)) = f(\Gamma) \),
2. \( \arg \min f(S, \tilde{\Gamma}(\Gamma)) \neq \emptyset \).

**Proof.** We proceed in several steps.

First, we prove that for each \( \Gamma \),

\[
f(\tilde{\Gamma}(\Gamma)) = f(\Gamma).
\]

(3.3)
If \( \Gamma(\Gamma) = \Gamma \), then this is trivial. Let \( \Gamma(\Gamma) \neq \Gamma \). Then there exists \( \tilde{S} \in \Omega(\Gamma) \) such that \((Dx, \tilde{S}x) < 0 \) on \( \Gamma \setminus \Gamma(\Gamma) \). Hence for each \( S \) we have the formula
\[
f(S, \Gamma) = \lim_{\lambda \to \infty} f(S + \lambda \tilde{S}, \Gamma),
\]
which, in turn, yields the inequality
\[
f(\Gamma(\Gamma)) \geq f(\Gamma).
\]
The reverse inequality is obvious. Thus, (3.3) is proved, which immediately leads to assertion (1) of the lemma.

We now prove assertion (2). If \( \Gamma(\Gamma) = \{0\} \), then this assertion is trivial since, by definition, \( f(S, \{0\}) = -\infty \) for each \( S \) and therefore \( \text{Argmin} \ f(\{0\}) = \emptyset \).

Before we consider the case of non-trivial \( \Gamma(\Gamma) \) we prove the equivalence
\[
\Gamma(\Gamma) = \Gamma \iff \Omega(\Gamma) = \{0\}. \tag{3.4}
\]
Indeed, if \( \Omega(\Gamma) = \{0\} \), then \( \Gamma(\Gamma) = \Gamma \) by definition. Conversely, if \( \Gamma(\Gamma) = \Gamma \), then, also by definition, \( \Omega(\Gamma) \subset \mathcal{J}_0(\Gamma) \), and therefore \( \Omega(\Gamma) = \{0\} \). The proof of (3.4) is thus complete.

Assume now that \( \Gamma(\Gamma) \neq \{0\} \). From \( \Gamma(\Gamma) = \Gamma(\Gamma) \) and (3.4) we obtain \( \Omega(\Gamma(\Gamma)) = \{0\} \). Let \( \{S_n\} \) be a minimizing sequence for \( f(\Gamma(\Gamma)) \). If \( \{S_n\} \) is bounded, then assertion (2) of the lemma holds. Let \( \{S_n\} \) be an unbounded sequence. Without loss of generality one can assume that \( S_n \in \mathcal{J}_1(\Gamma(\Gamma)) \) for each \( n \). This follows from an elementary property: if
\[
S = S' + S'', \quad S' \in \mathcal{J}_0(\Gamma(\Gamma)) \quad \text{and} \quad S'' \in \mathcal{J}_1(\Gamma(\Gamma)),
\]
then
\[
\omega(x, Sx) = \omega(x, S''x) \quad \text{on} \quad \Gamma.
\]
Further, also without loss of generality one can assume that \( \|S_n\| \to \infty \) and \( S_n/\|S_n\| \to \tilde{S} \). Obviously, \( \tilde{S} \in \mathcal{J}_1(\Gamma) \) and \( \|\tilde{S}\| = 1 \). However, we claim that
\[
\tilde{S} \in \Omega(\tilde{\Gamma}(\Gamma)). \tag{3.5}
\]
Indeed, if there exists an element \( x \in \tilde{\Gamma}(\Gamma) \) such that we have \( (E\tilde{S}x, \tilde{S}x) > 0 \), then \( f(S_n, \tilde{\Gamma}(\Gamma)) \to -\infty \) as \( n \to \infty \), which is impossible. Therefore, \( (E\tilde{S}x, \tilde{S}x) = 0 \) on \( \tilde{\Gamma}(\Gamma) \). Further, if there exists \( x \in \tilde{\Gamma}(\Gamma) \) such that \( (Dx, \tilde{S}x) > 0 \), then once more \( f(S_n, \tilde{\Gamma}(\Gamma)) \to -\infty \). Therefore, \( (Dx, \tilde{S}x) \leq 0 \) on \( \tilde{\Gamma}(\Gamma) \), and \( \tilde{S} \in \Omega(\tilde{\Gamma}(\Gamma)) \). The proof of (3.5) is thus complete. Hence \( \Omega(\tilde{\Gamma}(\Gamma)) \neq \{0\} \), and we arrive at a contradiction. Hence the sequence \( \{S_n\} \) is bounded, which completes the proof of the lemma.

We now return to the problem of § 1 and consider the quadratic form
\[
\omega(x, y) = (Ax + B^*y)^2 - 2Kxy - Qxx.
\]
It satisfies the conditions of Lemma 3. We have \( \omega(x, Sx) = -\omega_S(x) \).
Lemma 4. Assume that $\Omega(\mathbb{R}^d(x)) \neq \{0\}$. Then the system $\dot{x} = Kx + Bu$ is not controllable.

Proof. Consider the subspace $\Gamma_0 = \{y : B^*y = 0\}$. From the definition of $\Omega(\mathbb{R}^d(x))$ it follows that $Sx \in \Gamma_0$ for all $S \in \Omega(\mathbb{R}^d(x))$ and all $x \in \mathbb{R}^d(x)$. Hence the subspaces $\Gamma_0$ and $\Gamma'_0$ are $S$-invariant and

$$S\Gamma'_0 = \{0\},$$

where $\Gamma'_0$ is the orthogonal complement to $\Gamma_0$ in $\mathbb{R}^d(x)$. From the conditions of the lemma we obtain

$$\Gamma'_0 \neq \mathbb{R}^d(x).$$

Setting $\Gamma'_n = K^n\Gamma'_0$, $n = 1, 2, \ldots$ we claim that for each $n$,

$$Sx \perp \Gamma'_n$$

for all $S \in \Omega(\mathbb{R}^d(x))$ and $x \in \mathbb{R}^d(x)$. (3.8)

Indeed, (3.6) shows that (3.8) holds for $n = 0$. Assume that (3.8) holds for some $n$. Denoting by $\Gamma_n$ the orthogonal complement to $\Gamma'_n$ in $\mathbb{R}^d(x)$ we obtain

$$Sx \in \Gamma_n$$

for all $S \in \Omega(\mathbb{R}^d(x))$ and $x \in \mathbb{R}^d(x)$.

We denote by $\xi$ the elements of $\Gamma_n$, and by $\zeta$ the elements of $\Gamma'_n$. By the definition of the cone $\Omega(\mathbb{R}^d(x))$ the matrix $S$ satisfies the inequality

$$(Sx, Kx) \geq 0$$

for $x \in \mathbb{R}^d(x)$.

Then

$$(S\xi, K\xi) + (S\xi, K\zeta) = 0$$

for all $\xi, \zeta$,

which shows that

$$(S\xi, K\zeta) = 0$$

for all $\xi, \zeta$.

In view of the equality $S\Gamma'_n = 0$, this yields

$$(Sx, K\Gamma'_n) = 0$$

for each $x \in \mathbb{R}^d(x)$.

However, $K\Gamma'_n = \Gamma'_{n+1}$. We have thus proved that $Sx \perp \Gamma'_{n+1}$ for all $x$. This and the inductive assumption lead us to (3.8).

Consider the subspace $\overline{\Gamma}_n = \sum_{k=0}^n \Gamma'_k$, and let $\overline{\Gamma}_n$ be its orthogonal complement in $\mathbb{R}^d(x)$. Obviously, $\{\overline{\Gamma}_n\}$ is a non-decreasing sequence of subspaces, and therefore $\{\overline{\Gamma}_n\}$ is a non-increasing sequence.

We set $\overline{\Gamma}_\infty = \bigcap_n \overline{\Gamma}_n$ and prove that

$$\overline{\Gamma}_\infty \neq \{0\}. \quad (3.9)$$

Indeed, otherwise we see from (3.8) that $S = 0$, which contradicts the hypothesis of the lemma. The proof of (3.9) is thus complete. Denoting by $\overline{\Gamma}'_\infty$ the orthogonal complement to $\overline{\Gamma}_\infty$ in $\mathbb{R}^d(x)$ and using (3.8) we see that

$$K\overline{\Gamma}'_\infty \subset \overline{\Gamma}'_\infty. \quad (3.10)$$
Moreover, the embedding $\Gamma_\infty \subset \Gamma_0$ means that
\[ B^* \Gamma_\infty = \{0\}. \quad (3.11) \]

Denoting by $\xi$ elements of $\Gamma_\infty$, and by $\zeta$ elements of $\Gamma'_\infty$ we obtain
\[ \dot{\xi} = P_\infty K \xi + P_\infty K \zeta + P_\infty B u, \]
where $P_\infty$ is the orthogonal projection onto $\Gamma_\infty$. Then by (3.10) and (3.11) we obtain $\dot{\xi} = P_\infty K \xi$. This means, however, that the system $\dot{x} = K x + B u$ is uncontrollable, which completes the proof.

Thus, if the system $V$: $\dot{x} = K x + B u$ is controllable, then $\Omega(\mathbb{R}^d(x)) = \{0\}$ by Lemma 4, and therefore $\Gamma = \mathbb{R}^d(x)$ by Lemma 3. On the other hand, a controllable system $V$ can be decomposed into two systems, one that is a closed differential equation without control, and another that is controllable. Of course, one of these subsystems may be missing. The state components of the first subsystem vanish identically on $L(V)$ in view of the initial conditions. Hence we shall actually deal with the second, controllable subsystem. Thus, the assumption of controllability does not reduce the generality of our discussion.

§4. Proof of Theorem 1. Duality

4.1. First, we prove that the inequality $\lambda \geq 0$ yields the non-negativity of $J$ on $L(V)$. Applying Lemma 4 to the quadratic 2-form
\[ \omega(x, y) = (Ax + B^* y)^2 - 2Kxy - Qxx \]
and taking into account the controllability of system $V$ we obtain a matrix $\widetilde{S}$ such that
\[ \lambda(\widetilde{S}) = \lambda. \]
Then $\Phi(x, u) \geq 0$ for all $x, u$, which shows that $J \geq 0$ on $L(V)$.

Before considering the case $\lambda < 0$ we prove the following result. By an elementary cycle we shall mean a trajectory of the system $V$ of the following form:
\[ x(t) = x_0 \cos \sigma t + x_1 \sin \sigma t, \quad u(t) = u_0 \cos \sigma t + u_1 \sin \sigma t, \]
where $(x_0, x_1) \neq 0$ and $x_0 \perp x_1$. We denote by $T_\sigma$ an arbitrary period of an elementary cycle.

**Proposition 1.** (1) For an elementary cycle the following inequality holds:
\[ \frac{\int_0^{T_\sigma} \Phi(x(t), u(t)) dt}{\int_0^{T_\sigma} x^2(t) dt} \geq \lambda. \quad (*) \]

(2) There exists an elementary cycle turning $(*)$ into an equality.
Proof. Note that the ratio in the left-hand side of (*) does not depend on $T_\sigma$. Further, it is clear that
\[
\int_0^{T_\sigma} \Phi(x(t), u(t)) \, dt = \int_0^{T_\sigma} \Phi_S(x(t), u(t)) \, dt
\]
for an arbitrary elementary cycle, its arbitrary period $T_\sigma$, and each $S$. Hence
\[
\frac{\int_0^{T_\sigma} \Phi(x(t), u(t)) \, dt}{\int_0^{T_\sigma} x^2(t) \, dt} \geq \frac{\int_0^{T_\sigma} \omega_S(x) \, dt}{\int_0^{T_\sigma} x^2(t) \, dt}.
\]
However, $\omega_S(x) \geq \hat{\lambda}x^2$, which proves assertion (1) of Proposition 1.

We shall use the relation
\[
\lambda(\hat{S}) = \max_S \lambda(S).
\]
The matrix $\hat{S}$ must satisfy the following necessary condition:
\[
\min_{x \in \Sigma_0} (R_S x, \overline{S} x) \leq 0 \quad \text{for each } \overline{S},
\]
where $R_S = K + B U_S$,
\[
\Gamma_0 = \{ x : \omega_S(x) = \hat{\lambda}x^2 \},
\]
\[
\Sigma_0 = \{ x \in \Gamma_0 : |x| = 1 \}.
\]
In other words, the matrix $-R_S$ satisfies the condition $(\ast; \Gamma_0)$; see § 3. By Lemma 2 there exists a subspace $\Gamma_1$ such that
\[
\Gamma_1 \subset \Gamma_0, \quad d(\Gamma_1) > 0, \quad R_S \Gamma_1 \subset \Gamma_1,
\]
and $R_S$ has an eigenvalue with real part zero on $\Gamma_1$. Hence there exists an $R_S$-invariant subspace $\Gamma$ of $\Gamma_1$ of dimension $d(\Gamma) = 1$ or 2. Thus $\Gamma$ is the eigensubspace of $R_S$ corresponding to this eigenvalue. If $d(\Gamma) = 1$, then the eigenvalue of $R_S$ on $\Gamma$ is zero, and $R_S \Gamma = 0$. If $d(\Gamma) = 2$, then $R_S$ has a pair of complex conjugate eigenvalues $\pm i\sigma$, $\sigma \neq 0$, on $\Gamma$ and the operator $R_S$ can be represented in the following form:
\[
R_S x_0 = \sigma x_1, \quad R_S x_1 = -\sigma x_0,
\]
where
\[
\text{Lin}(x_0, x_1) = \Gamma, \quad (x_0, x_1) = 0.
\]

(4.1)

For definiteness we consider the last (two-dimensional) case. Let $x(t)$ be a solution to the equation
\[
\frac{dx}{dt} = R_S x, \quad x(0) = x_0.
\]
Setting \( u(t) = U_S x(t) \) we see that the pair \( x(t), u(t) \) is an elementary cycle representable in the following form:

\[
\begin{align*}
  x(t) &= x_0 \cos \sigma t + x_1 \sin \sigma t, \\
  u(t) &= u_0 \cos \sigma t + u_1 \sin \sigma t,
\end{align*}
\]

where \( x_0 \) and \( x_1 \) are defined in (4.1), \( u_0 = U_S x_0 \), and \( u_1 = U_S x_1 \). Then

\[
\Phi_S (x(t), u(t)) = \omega_S (x(t)) = \hat{\lambda} (x(t))^2,
\]

which yields assertion 2. This completes the proof of Proposition 1.

We continue the proof of Theorem 1 and consider the case \( \hat{\lambda} < 0 \).
Let \( x(t), u(t) \) be an elementary cycle such that

\[
\int_0^{T_\sigma} \Phi(x(t), u(t)) \, dt = A.
\]

Since the system \( V \) is controllable, it is possible to proceed from the origin to the state component of the cycle in finite time, stay there for arbitrarily large time equal to some period \( T_\sigma \), and return to the origin, also in finite time. For sufficiently large \( T_\sigma \) we obtain a trajectory in \( L(V) \) such that \( J < 0 \). This proves Theorem 1.

4.2. Proposition 1 establishes a connection between the frequency criterion and the criterion based on the Hamilton–Jacobi inequality. This connection itself is a kind of duality theorem.

Let \( x(t), u(t) \) be a periodic trajectory of the system \( V \), and let \( \{T\} \) be the set of its periods. Consider the functional

\[
Z(\Phi) = \frac{\int_0^T \Phi(x(t), u(t)) \, dt}{\int_0^T x^2(t) \, dt}, \quad T \in \{T\}.
\]

Obviously, \( Z(\Phi) \) does not depend on the choice of \( T \) and is a linear functional on the set of functions \( \Phi(x, u) \) of the form considered in § 1, and this set — we denote it by \( \tilde{\Phi} \) — is convex. Thus, each non-trivial cycle gives rise to a linear functional on the convex set \( \tilde{\Phi} \). We consider now another functional on \( \tilde{\Phi} \):

\[
Y(\Phi) = \min \left\{ \frac{\Phi(x, u)}{x^2} : u \in \mathbb{R}^d(u), \ x \in \mathbb{R}^d(x) \setminus \{0\} \right\}.
\]

Note that the inclusion \( \Phi \in \tilde{\Phi} \) means that \( \Phi_S \in \tilde{\Phi} \) for each \( S \). Moreover, if \( S = 0 \), then \( \Phi_S = \Phi \). We denote by \( \zeta \) an arbitrary non-trivial periodic trajectory of the system \( V \).
Proposition 2. The following duality formula holds:

\[
\min_{\zeta} Z(\Phi) = \max_{S} Y(\Phi_{S}) \quad \text{for any } \Phi \in \tilde{\Phi}. \tag{4.2}
\]

Proof. In accordance with § 2 we have

\[
\Phi_{S}(x, u) = (u - U_{S}x)^{2} + \omega_{S}(x).
\]

Consequently, \(Z(\Phi_{S}) \geq \lambda(S)\) for each \(\zeta\). Hence, in view of the equality \(Z(\Phi_{S}) = Z(\Phi)\), we have

\[
Z(\Phi) \geq \lambda(S).
\]

Therefore,

\[
Z(\Phi) \geq \tilde{\lambda} \quad \text{for any } \zeta.
\]

On the other hand \(Y(\Phi_{S}) = \lambda(S)\), so that \(\tilde{\lambda} = \max_{S} Y(\Phi_{S})\). We have thus proved the inequality

\[
\min_{\zeta} Z(\Phi) \geq \max_{S} Y(\Phi_{S}). \tag{4.3}
\]

By Proposition 1 there exists an elementary cycle \(\zeta^{0}\) such that \(Z(\Phi) = \tilde{\lambda}\), which yields equality (4.2) and completes the proof of Proposition 2.

Consider now the frequency criterion [1]. We set

\[
f = u_{0}^{2} + u_{1}^{2} + 2(Ax_{0}u_{0} + Ax_{1}u_{1}) + Qx_{0}x_{0} + Qx_{1}x_{1}.
\]

The frequency criterion reads as follows. The inequality \(J \geq 0\) on \(L(V)\) holds if and only if \(f \geq 0\) for all \(\sigma, x_{0}, x_{1}, u_{0}, u_{1}\) such that

\[
\begin{align*}
\sigma x_{1} &= Kx_{0} + Bu_{0}, \\
-\sigma x_{0} &= Kx_{1} + Bu_{1}.
\end{align*} \tag{4.4}
\]

Proposition 1 can be written in the following form:

\[
\min_{\sigma} \frac{f}{x_{0}^{2} + x_{1}^{2}} = \tilde{\lambda}, \tag{4.5}
\]

where the minimum is taken over all \(\sigma, x_{0}, x_{1}, u_{0}, u_{1}\) satisfying (4.4). Indeed, the following relation holds for an arbitrary elementary cycle:

\[
\frac{f}{x_{0}^{2} + x_{1}^{2}} = \frac{\int_{0}^{T_{\sigma}} \Phi(x(t), u(t)) \, dt}{\int_{0}^{T_{\sigma}} x^{2}(t) \, dt} \quad \text{for each } T_{\sigma}.
\]

Combined with Proposition 1, this easily yields (4.5). Thus, the frequency criterion is a consequence of Theorem 1 and is not the only tool for the investigation of quadratic forms with constant coefficients on the half-axis.

In the next two sections we give examples of applications of Theorem 1 and Proposition 1.
§ 5. Quadratic regulation

5.1. Consider a control system $V: \dot{x} = Kx + Bu$ that is controllable in the sense of § 3. We set $\mathcal{U} = \text{pr}_{u(\cdot)} L(V)$ and $\mathcal{X} = \text{pr}_{x(\cdot)} L(V)$.

We equip the linear varieties $\mathcal{U}$ and $\mathcal{X}$ with the norms

$$
\|u(\cdot)\| = \sqrt{\int_{0}^{\infty} u^2(t) \, dt} \quad \text{and} \quad \|x(\cdot)\| = \sqrt{\int_{0}^{\infty} x^2(t) \, dt}.
$$

We consider now the operator

$$
F: \mathcal{U} \to \mathcal{X}, \quad u(\cdot) \to x(\cdot), \quad \text{where} \quad (u(t), x(t)) \in L(V),
$$

and pose the question: when is the operator $F$ bounded, and what is its norm?

Let $\Phi = u^2$. By § 2, for each $S$ we have

$$
\int_{0}^{\infty} \Phi(\cdot) \, dt = \int_{0}^{\infty} \left[ (u - USx)^2 + \omega_S(x) \right] \, dt.
$$

We set $\hat{\lambda} = \max_S \lambda(S)$. Since $\lambda(S) = 0$ for $S = 0$, it follows that $\hat{\lambda} \geq 0$.

**Theorem 2.** The operator $F$ is bounded if and only if $\hat{\lambda} > 0$. Its norm is $1/\sqrt{\hat{\lambda}}$.

**Proof.** Let $\hat{\lambda} > 0$ and let $\hat{S}$ be a matrix such that $\lambda(\hat{S}) = \hat{\lambda}$. We have

$$
\Phi_{\hat{S}}(x, u) - \hat{\lambda}x^2 = (u - U_{\hat{S}}x)^2 + \omega_{\hat{S}}(x) - \hat{\lambda}x^2 \geq 0.
$$

Hence

$$
J_{\hat{S}}(x(\cdot), u(\cdot)) \geq \hat{\lambda} \int_{0}^{\infty} x^2(t) \, dt \quad \text{on} \quad L(V).
$$

However, $J_{\hat{S}} = J$ on $L(V)$, therefore,

$$
\int_{0}^{\infty} u^2(t) \, dt \geq \hat{\lambda} \int_{0}^{\infty} x^2(t) \, dt,
$$

and $\|F\| \leq 1/\sqrt{\hat{\lambda}}$. On the other hand, by Proposition 1 there exists an elementary cycle $x^0(t)$, $u^0(t)$ such that

$$
\int_{0}^{T_{\pi}} (u^0(t))^2 \, dt = \hat{\lambda} \int_{0}^{T_{\pi}} (x^0(t))^2 \, dt,
$$

and therefore $\|F\| \geq 1/\sqrt{\hat{\lambda}}$. Hence

$$
\hat{\lambda} > 0 \Rightarrow \|F\| = \frac{1}{\sqrt{\hat{\lambda}}}. \quad (5.1)
$$

Let $\hat{\lambda} = 0$. By Proposition 1, there exists an elementary cycle $x^0(t)$, $u^0(t)$ such that $u^0(t) = 0$. This shows that $F$ is unbounded. The proof is complete.
We now have \( \omega_S(x) = -(B^*Sx)^2 + 2(Kx, Sx) \). Hence \( \tilde{\lambda} > 0 \) if and only if there exists \( S \) such that
\[
(Kx, Sx) > 0 \quad \text{for each } x \neq 0.
\]
By Lemma 1 this is equivalent to the condition that \( K \) has no eigenvalues with real part zero. Thus, the operator \( F \) is bounded if and only if \( K \) has no eigenvalues with real part zero.

The closure of \( U \) in the norm of \( L_2 \) is a subspace of a finite codimension equal to the number of eigenvalues of \( K \) with negative real parts. If there exist no such eigenvalues, then the closure of \( U \) is the whole of the space. Hence the solution \( x(t) \) to the equation \( \dot{x} = Kx + Bu, \ x(0) = 0 \), belongs to \( L_2 \) for each \( u(t) \in L_2 \) if and only if all the eigenvalues of \( K \) have negative real parts.

5.2. Let \( \Phi \) and \( V \) be fixed. Consider the following problem:

\[
J = \int_0^\infty \Phi \ dt \to \inf, \quad \dot{x} = Kx + Bu, \quad x(0) = x_*, \quad \int_0^\infty (x^2 + u^2) \ dt < +\infty. \tag{5.2}
\]

We assume as before that the system \( V \) is controllable.

The following classification is presented in the review paper [1]. Three distinct cases are possible:

(1) \( \inf J = -\infty \) for all \( x_* \);
(2) \( \inf J = \min J \) for all \( x_* \) and \( \text{Argmin} \ J \) consists of one point (the regular case);
(3) \( \inf J > -\infty \) for all \( x_* \), but the case is not regular.

We shall prove this classification making no use of the frequency criterion.

**Theorem 3.** The case when \( \inf J = -\infty \) for all \( x_* \) is equivalent to the inequality \( \tilde{\lambda} < 0 \).

The case when \( \text{Argmin} \ J \neq \emptyset \) for all \( x_* \) is equivalent to the inequality \( \tilde{\lambda} > 0 \).

The case when \( \inf J > -\infty \) for all \( x_* \) but there exists \( x_* \) such that \( \text{Argmin} \ J = \emptyset \) is equivalent to the equality \( \tilde{\lambda} = 0 \).

**Proof.** Assume that \( \tilde{\lambda} < 0 \). Then, in accordance with Theorem 1 there exists a trajectory \( (x^0(t), u^0(t)) \in L(V) \) such that
\[
J(x^0(\cdot), u^0(\cdot)) < 0.
\]

For an arbitrary trajectory \( (x^1(t), u^1(t)) \) admissible in the problem (5.2) the trajectory \( (x^1(t) + \tau x^0(t), u^1(t) + \tau u^0(t)) \) is also admissible for each \( \tau \), and we have \( J(x^1 + \tau x^0, u^1 + \tau u^0) \to -\infty \) as \( \tau \to \infty \). Hence \( \inf J = -\infty \) for all \( x_* \).

Assume that \( \tilde{\lambda} > 0 \). We choose \( \tilde{S} \) such that \( \lambda(\tilde{S}) = \tilde{\lambda} \) and set
\[
J_{\tilde{S}} = \int_0^\infty \left( \Phi + \frac{d}{dt}(\tilde{S}xx) \right) \ dt = J - (\tilde{S}x_*x_*). \tag{5.3}
\]
We have

\[ J_\delta = \int_0^\infty [(u - U_\delta x)^2 + \omega_\delta(x)] \, dt \geq \int_0^\infty [(u - U_\delta x)^2 + \lambda x^2] \, dt \]

for any admissible trajectory in the problem (5.2). Hence there exists \( \delta > 0 \) such that

\[ J_\delta \geq \delta \int_0^\infty [x^2(t) + u^2(t)] \, dt. \]

Then the set \( \text{Argmin} J_\delta \) is non-empty for all \( x_* \) and consists of a single element. However, \( \text{Arg min} J_\delta = \text{Arg min} J \).

Let \( \lambda = 0 \). Then \( J_\delta \geq 0 \) for any admissible trajectory, and therefore \( \inf J > -\infty \) for all \( x_* \). By Proposition 1 there exists an elementary cycle such that \( \int_0^T \Phi \, dt = 0 \) for each period \( T \). We denote this cycle by \( x^0(t), u^0(t) \). Let \( x_* = x^0(0) \). Assume that \( \text{Arg min} J \neq \emptyset \) for this vector \( x_* \). Then \( \text{Arg min} J \) contains more than one trajectory. However, it can easily be shown that if \( \text{Arg min} J \neq \emptyset \), then this set consists of one element. For let

\[ (x^1(t), u^1(t)) \in \text{Arg min} J \quad \text{and} \quad (x^2(t), u^2(t)) \in \text{Arg min} J. \]

Since \( \lambda = 0 \), it follows that \( J_\delta \) is a convex functional over the admissible trajectories of the problem (5.2). We set \( x^1 - x^2 = \delta x, \quad u^1 - u^2 = \delta u \). Then the following equality must hold:

\[ J_\delta(\delta x(\cdot), \delta u(\cdot)) = 0. \]

Hence \( \delta u = U_\delta \delta x \), and therefore \( \delta x = 0, \delta u = 0 \) by virtue of the system \( V \). Hence \( \text{Arg min} J \) contains at most one admissible trajectory. Then, however, \( \text{Arg min} J = \emptyset \) for the \( x_* \) in question. The proof is complete.

\[ \text{§ 6. Necessary and sufficient conditions resulting from Theorem 1 for the non-negativity of the form } J \text{ on } L(V) \]

6.1. Necessary and sufficient conditions for the non-negativity of the form \( J \) on \( L(V) \) will be briefly called \textit{non-negativity conditions}. The following non-negativity conditions follow directly from the proof of Theorem 1.

A pair \( \Gamma, S \) will be called \textit{compatible} if it satisfies the following conditions:

\[ d(\Gamma) \in \{1, 2\}; \quad (6.11) \]

\[ R_S \Gamma \subset \Gamma; \quad (6.12) \]

the eigenvalues of \( R_S \) on \( \Gamma \) are equal to 0 if \( d(\Gamma) = 1 \)

and to \( \pm i\sigma, \sigma \neq 0 \) if \( d(\Gamma) = 2 \); \quad (6.13)

\[ \omega_S(x) = \kappa x^2 \text{ on } \Gamma; \quad (6.14) \]

\[ \omega'_S(x, x') = 0 \text{ for all } x \in \Gamma, \ x' \perp \Gamma. \quad (6.15) \]

Then it is obvious from the proof of Theorem 1 that the following condition is a non-negativity condition.
Condition 1. For each compatible pair $\kappa \geq 0$.

A pair $\Gamma, S$ is said to be weakly compatible if it satisfies conditions (6.11)–(6.14). The following is also a non-negativity condition.

Condition 2. For each weakly compatible pair $\kappa \geq 0$.

Similarly to Condition 1 this also follows directly from the proof of Theorem 1. We ‘decipher’ Conditions 1 and 2 below.

6.2. In this section we prove the formula

$$\frac{1}{2} \omega'_s(x, y) = (U_s x, A y) + (SR_s x, y) + (S x, K y) + (Q x, y) \quad (6.2)$$

for all $x$ and $y$. Here, by $\omega'_s(x, y)$ we denote the derivative of the form $\omega_s$ at the point $x$ in the direction $y$:

$$\omega'_s(x, y) = \frac{d}{d\varepsilon} \omega_s(x + \varepsilon y) \bigg|_{\varepsilon = 0}$$

By § 2 we obtain

$$\omega_s(x) = -(U_s x)^2 + 2(K x, S x) + (Q x, x).$$

Therefore,

$$\frac{1}{2} \omega'_s(x, y) = -(U_s x, U_s y) + (K x, S y) + (S x, K y) + (Q x, y). \quad (6.3)$$

By definition $U_s = -(A + B^* S)$. Hence

$$-(U_s x, U_s y) = (U_s x, A y) + (U_s x, B^* S y).$$

On the other hand,

$$(K x, S y) = (SR_s x, y) - (SBU_s x, y),$$

which follows from the definition of $R_s$: $R_s = K + BU_s$. In view of the last two equalities,

$$\frac{1}{2} \omega'_s(x, y) = (U_s x, A y) + (SR_s x, y) + (S x, K y) + (Q x, y),$$

as required.

6.3. Let $d(\Gamma) = 1$. We now find conditions for the existence of a matrix $S$ weakly compatible with $\Gamma$. 
Theorem 4. A matrix $S$ weakly compatible with $\Gamma$ exists if and only if there exist $\hat{x}$, $\hat{p}$, and $\hat{u}$ such that

1. $\hat{x} \in \Gamma$, $\hat{x} \neq 0$;
2. $0 = K\hat{x} + B\hat{u}$;
3. $\hat{u} = -(A\hat{x} + B^*\hat{p})$.

Proof. Let $S$ be a matrix such that the pair $\Gamma$, $S$ is weakly compatible. We fix arbitrary $\hat{x} \in \Gamma$, $\hat{x} \neq 0$, and set $\hat{p} = S\hat{x}$. By the definition of weak compatibility $R_S\hat{x} = 0$. Hence, setting $\hat{u} = -(A\hat{x} + B^*\hat{p})$ we obtain $0 = K\hat{x} + B\hat{u}$. This proves the necessity of the conditions in the theorem.

Assume now that there exist $\hat{x}$, $\hat{p}$, and $\hat{u}$, satisfying conditions (1)–(3) in the theorem. We fix $S$ such that $S\hat{x} = \hat{p}$. Then, by condition (3),

$$
\hat{u} = U_S\hat{x},
$$

$$
K\hat{x} + B\hat{u} = R_S\hat{x}.
$$

By condition (2) this means that $R_S\hat{x} = 0$. The proof of the sufficiency of conditions (1)–(3) in the theorem and of the entire theorem is complete.

Next, we find conditions for the existence of $S$ compatible with $\Gamma$.

Theorem 5. A matrix $S$ compatible with $\Gamma$ exists if and only if there exist $\hat{x}$, $\hat{p}$, and $\hat{u}$ such that

1. $\hat{x} \in \Gamma$, $\hat{x} \neq 0$;
2. $0 = K\hat{x} + B\hat{u}$;
3. $\hat{u} = -(A\hat{x} + B^*\hat{p})$;
4. $\hat{u}A + \hat{p}K + Q\hat{x} = K\hat{x}$.

Proof. Let $S$ be compatible with $\Gamma$. We fix arbitrary $\hat{x} \in \Gamma$, $\hat{x} \neq 0$, and set $\hat{p} = S\hat{x}$, $\hat{u} = U_S\hat{x}$. Obviously, $\hat{x}$, $\hat{p}$, and $\hat{u}$ satisfy conditions (1)–(3) in the theorem since $R_S\hat{x} = 0$ by (6.13). Further, by (6.15) we obtain

$$
\omega_S'(\hat{x}, x') = 0 \quad \text{for all } x' \perp \hat{x}.
$$

By formula (6.2) this means that

$$
(\hat{u}, Ax') + (\hat{p}, Kx') + (Q\hat{x}, x') = 0 \quad \text{for all } x' \perp \hat{x},
$$

which yields condition (4) in the theorem and proves the necessity.

We now proceed to the proof of the sufficiency. Assume that $\hat{x}$, $\hat{p}$, and $\hat{u}$ satisfy conditions (1)–(4) in the theorem. We fix $S$ such that $S\hat{x} = \hat{p}$. We claim that $S$ is compatible with $\Gamma$. Using the same arguments as in the proof of Theorem 4 we see that $S$ is weakly compatible with $\Gamma$. We now prove (6.15). By formula (6.2), for $x' \perp \hat{x}$ we have

$$
\frac{1}{2} \omega_S'(\hat{x}, x') = (\hat{u}, Ax') + (SR_S\hat{x}, x') + (\hat{p}, Kx') + (Q\hat{x}, x') = \kappa(\hat{x}, x') = 0.
$$

The second equality follows from condition (4) in the theorem. We have thus established equality (6.15). The proof of the sufficiency and of the entire theorem is complete.
6.4. In this subsection we discuss the same question as in §6.3, but for two-dimensional \( \Gamma \). Thus, let \( \Gamma \) be a fixed subspace of dimension \( d(\Gamma) = 2 \).

**Theorem 6.** A matrix \( S \) weakly compatible with \( \Gamma \) exists if and only if there exist \( \sigma, \kappa; x_0, x_1; p_0, p_1; u_0, u_1 \) such that

1. \( \sigma \neq 0 \);
2. \( \text{Lin}(x_0, x_1) = \Gamma, \quad (x_0, x_1) = 0 \);
3. \( p_1x_0 = p_0x_1 \);
4. \( \sigma x_1 = Kx_0 + Bu_0, \quad -\sigma x_0 = Kx_1 + Bu_1 \);
5. \( u_0 = -(Ax_0 + B^*p_0), \quad u_1 = -(Ax_1 + B^*p_1) \);
6. \( -u_0^2 + 2(p_0, Kx_0) + (Qx_0, x_0) = \kappa x_0^2, \)
   \( -u_1^2 + 2(p_1, Kx_1) + (Qx_1, x_1) = \kappa x_1^2, \)
   \( -u_0u_1 + (p_1, Kx_0) + (p_0, Kx_1) + (Qx_0, x_1) = 0. \)

**Proof.** Let \( S \) be a matrix weakly compatible with \( \Gamma \). By (6.12), the operator \( R_S \) has an invariant subspace \( \Gamma \), and by (6.13) its eigenvalues on \( \Gamma \) are \( \pm i\sigma \) with \( \sigma \neq 0 \). This determines the constant \( \sigma \). Further, there exists an orthonormal basis \( x_0, x_1 \) in \( \Gamma \) such that the operator \( R_S \) has the following representation in this basis:

\[
\sigma x_1 = R_Sx_0, \quad -\sigma x_0 = R_Sx_1.
\]

These relations determine the vectors \( x_0 \) and \( x_1 \). Note that \( \sigma, x_0, \) and \( x_1 \) satisfy conditions (1) and (2) in the theorem. Further, let \( p_0 = Sx_0, \quad p_1 = Sx_1, \quad u_0 = U_Sx_0, \quad u_1 = U_Sx_1 \).

Using the definitions of \( U_S \) and \( R_S \) it is easy to show that \( \sigma, x_0, x_1; p_0, p_1; u_0, u_1 \) satisfy conditions (3)–(5) in the theorem. It remains to demonstrate that conditions (6) are also satisfied.

By (6.14), \( \omega_S(x) = \kappa x^2 \) on \( \Gamma \). This determines the constant \( \kappa \). Equality (6.14) is obviously equivalent to the relations

\[
\omega_S(x_0) = \kappa x_0^2, \quad \omega_S(x_1) = \kappa x_1^2, \quad \omega'_S(x_0, x_1) = 0 \quad \text{(since} \ x_0 \perp x_1 \text{)},
\]

which easily yield conditions (6) in the theorem and complete the proof of the necessity.

We now proceed to the sufficiency. Assume that \( \sigma, \kappa; x_0, x_1; p_0, p_1; u_0, u_1 \) satisfy conditions (1)–(6) in the theorem. Consider a matrix \( S \) such that \( Sx_0 = p_0 \) and \( Sx_1 = p_1 \). These requirements are consistent in view of condition (3). Then \( u_0 = U_Sx_0 \) and \( u_1 = U_Sx_1 \). Conditions (4) and (5) yield

\[
\sigma x_1 = R_Sx_0, \quad -\sigma x_0 = R_Sx_1.
\]

Hence \( \Gamma \) is \( R_S \)-invariant and the eigenvalues of \( R_S \) on \( \Gamma \) are equal to \( \pm i\sigma \). Condition (6) shows that

\[
\omega_S(x) = \kappa x^2 \quad \text{on} \quad \Gamma.
\]

The proof of the sufficiency and the entire proof of Theorem 6 are complete.

We now find conditions for the existence of \( S \) compatible with \( \Gamma \).
Theorem 7. A matrix $S$ compatible with $\Gamma$ exists if and only if there exist $\sigma$, $\kappa$; $x_0$, $x_1$; $p_0$, $p_1$; $u_0$, $u_1$ such that

1. $\sigma \neq 0$;
2. $\text{Lin}(x_0, x_1) = \Gamma$, $(x_0, x_1) = 0$;
3. $p_1 x_0 = p_0 x_1$;
4. $\sigma x_1 = K x_0 + B u_0$, $-\sigma x_0 = K x_1 + B u_1$;
5. $u_0 = -(A x_0 + B^* p_0)$, $u_1 = -(A x_1 + B^* p_1)$;
6. $u_0 A + \sigma p_1 + p_0 K + Q x_0 = \kappa x_0$,

$u_1 A - \sigma p_0 + p_1 K + Q x_1 = \kappa x_1$.

Proof. We prove the necessity. Let $S$ be a matrix compatible with $\Gamma$. Then $S$ is weakly compatible with $\Gamma$ and, as in the proof of Theorem 6, we can find $\sigma$, $\kappa$; $x_0$, $x_1$; $p_0$, $p_1$; $u_0$, $u_1$, satisfying all the conditions in Theorem 6. From (6.1) we obtain

$$\omega_S'(x_0, x') = 0, \quad \omega_S'(x_1, x') = 0 \quad \forall x' \perp \Gamma.$$ 

The conditions in Theorem 6 show that for $x' \in \Gamma$,

$$u_0 A x' + \sigma p_1 x' + (p_0, K x') + (Q x_0, x') = 0,$$

$$u_1 A x' - \sigma p_0 x' + (p_1, K x') + (Q x_1, x') = 0.$$ 

Therefore,

$$u_0 A + \sigma p_1 + p_0 K + Q x_0 \in \Gamma,$$

$$u_1 A - \sigma p_0 + p_1 K + Q x_1 \in \Gamma. \quad (6.4)$$ 

Multiplying the left-hand side of the first inclusion by $x_1$ and using formula (6.2) we obtain

$$\frac{1}{2} \omega_S'(x_0, x_1) = u_0 A x_1 + \sigma p_1 x_1 + p_0 K x_1 + Q x_0 x_1.$$ 

However,

$$u_0 A x_1 = -u_0 u_1 - u_0 B^* p_1,$$

$$\sigma p_1 x_1 = p_1 K x_0 + p_1 B u_0,$$

so that

$$\frac{1}{2} \omega_S'(x_0, x_1) = -u_0 u_1 + p_1 K x_0 + p_0 K x_1 + Q x_0 x_1.$$ 

It follows now by conditions (6) in Theorem 6 that $\frac{1}{2} \omega_S'(x_0, x_1) = 0$. Then, however, (6.4) yields

$$u_0 A + \sigma p_1 + p_0 K + Q x_0 = \kappa_0 x_0,$$

$$u_1 A - \sigma p_0 + p_1 K + Q x_1 = \kappa_1 x_1.$$ 

Multiplying the first equality by $x_0$ we obtain

$$\omega_S(x_0) = u_0 A x_0 + \sigma p_1 x_0 + p_0 K x_0 + Q x_0 x_0 = \kappa_0 x_0^2.$$
However,

\[ u_0Ax_0 = -u_0^2 - u_0B^*p_0, \]
\[ \sigma p_1 x_0 = \sigma p_0 x_1 = p_0Kx_0 + p_0Bu_0, \]

and therefore

\[ \omega_S(x_0) = -u_0^2 + 2p_0Kx_0 + Qx_0x_0, \]

which, in view of conditions (6) in Theorem 6, shows that \( \omega_S(x_0) = \kappa x_0^2 \). Thus, \( \kappa_0 = \kappa \). Using similar arguments one can prove that \( \kappa_1 = \kappa \), which proves the necessity.

We now prove the sufficiency. Let \( \sigma, \kappa; x_0, x_1; p_0, p_1; u_0, u_1 \) be a collection satisfying all the conditions in Theorem 7. We choose a matrix \( S \) such that

\[ Sx_0 = p_0, \quad Sx_1 = p_1. \]

By condition (3) of the theorem, such a choice is possible. Apart from this, \( S \) can be an arbitrary symmetric matrix. We shall show that \( S \) is compatible with \( \Gamma \). The assumptions of the theorem easily yield

\[ U_Sx_0 = u_0, \quad U_Sx_1 = u_1, \]
\[ R_Sx_0 = \sigma x_1, \quad R_Sx_1 = -\sigma x_0. \]

Hence conditions (6.11)–(6.13) are fulfilled. Conditions (6.14) and (6.15) follow easily from conditions (6) in the theorem. We do not discuss this in detail because similar considerations were made in the proof of the necessity. The proof of the sufficiency, and therefore of Theorem 7 in its entirety is complete.

We note here a fact which follows from the proofs of Theorems 4–7: if a pair \( \Gamma, S \) is compatible (weakly compatible), then the pair \( \Gamma, S' \) possesses the same property for any matrix \( S \) coinciding with \( S \) on \( \Gamma \).

Conditions 1 and 2 now take the following form. For each set of parameters satisfying the conditions in one of Theorems 4–7, the inequality \( \kappa \geq 0 \) must hold. Perhaps, it is possible to obtain non-negativity conditions that also involve in an essential fashion some properties of the matrix \( S \) outside \( \Gamma \).

6.5. In this subsection we consider a control system \( V \) of the form \( \dot{x} = u \). Here

\[ U_S = R_S = -(A + S), \]
\[ \omega_S(x) = -(U_Sx)^2 + Qxx. \]  \hspace{1cm} (6.5)

In this case the matrix \( A \) can — and will — be assumed to be skew-symmetric. It is also natural to assume that the matrix \( Q \) is non-negative definite. Then conditions 1 and 2 can be left out for one-dimensional \( \Gamma \). We shall consider a non-negativity condition based on Theorem 6. In our case it reads as follows.
For any $\sigma, \kappa; x_0, x_1; p_0, p_1; u_0, u_1$, satisfying the conditions

\begin{align*}
\sigma &\neq 0, \\
d(\text{Lin}(x_0, x_1)) &= 2, \quad (x_0, x_1) = 0, \\
p_1 x_0 &= p_0 x_1, \\
\sigma x_1 &= u_0, \quad -\sigma x_0 = u_1, \\
u_0 &= -(Ax_0 + p_0), \quad u_1 = -(Ax_1 + p_1), \\
-u_0^2 + (Qx_0, x_0) &= \kappa x_0^2, \quad -u_1^2 + (Qx_1, x_1) = \kappa x_1^2, \quad (Qx_0, x_1) = 0,
\end{align*}

the inequality

$$\kappa \geq 0 \quad (6.7)$$

must hold. Note that (6.62) and (6.64) mean that $(u_0, u_1) = 0$. One can easily eliminate $u_0$ and $u_1$ from conditions (6.61)-(6.66), and express $\sigma$ in terms of $x_0, x_1$. From conditions (6.64) and (6.65) we obtain

$$\begin{align*}
\sigma x_1 &= -Ax_0 - p_0, \\
-\sigma x_0 &= -Ax_1 - p_1.
\end{align*}$$

Multiplying the first equality by $x_1$, the second by $x_0$, subtracting one from the other, and taking (6.63) into account we obtain

$$
\sigma(x_0^2 + x_1^2) = 2Ax_1 x_0. \quad (6.8)
$$

Then conditions (6.66) take the following form:

$$\begin{align*}
-\sigma^2 x_1^2 + Qx_0 x_0 &= \kappa x_0^2, \\
-\sigma^2 x_0^2 + Qx_1 x_1 &= \kappa x_1^2, \\
Qx_0 x_1 &= 0.
\end{align*} \quad (6.9)$$

One must add to this (6.61) and (6.62). Let us introduce further notation:

\begin{align*}
\bar{x}_0 &= \frac{x_0}{|x_0|}, \quad \bar{x}_1 = \frac{x_1}{|x_1|}, \\
\mu &= \frac{x_0^2}{x_1^2}, \quad a = 4(A\bar{x}, \bar{x}_0)^2, \\
q_0 &= (Q\bar{x}_0, \bar{x}_0), \quad q_1 = (Q\bar{x}_1, \bar{x}_1).
\end{align*}

In this notation conditions (6.9) take the following form:

$$\begin{align*}
-\frac{a}{(1+\mu)^2} + q_0 &= \kappa, \\
-\frac{a}{(1+1/\mu)^2} + q_1 &= \kappa, \\
a &> 0, \quad 0 < \mu < +\infty. \quad (6.10)
\end{align*}$$
We shall find out when conditions (6.10) are compatible and obtain an expression for $\mu$. From the relation

$$q_0 - q_1 = a \left( \frac{1}{(1 + \mu)^2} - \frac{1}{(1 + 1/\mu)^2} \right) = a \frac{1 - \mu^2}{(1 + \mu)^2} = a \frac{1 - \mu}{1 + \mu}$$

we see that

$$\mu = \frac{a - \Delta q}{a + \Delta q}, \quad \Delta q = q_0 - q_1. \tag{6.11}$$

Therefore, conditions (6.10) are compatible if and only if

$$a > |\Delta q|. \tag{6.12}$$

We now find out conditions equivalent to the inequality $\kappa \geq 0$. It turns out that these conditions can be formulated in terms of just $a$ and $q$.

Indeed, assume that $\kappa \geq 0$. Then by (6.10),

$$q_0 \geq \frac{a}{(1 + \mu)^2}, \quad q_1 \geq \frac{a}{(1 + 1/\mu)^2},$$

and therefore

$$\sqrt{q_0} \geq \frac{\sqrt{a}}{1 + \mu}, \quad \sqrt{q_1} \geq \frac{\sqrt{a}}{1 + 1/\mu}.$$  

On summing we obtain $\sqrt{q_0} + \sqrt{q_1} \geq \sqrt{a}$.

Assume now that $\kappa < 0$. Then

$$\sqrt{q_0} < \frac{a}{1 + \mu}, \quad \sqrt{q_1} < \frac{a}{1 + 1/\mu},$$

and therefore $\sqrt{q_0} + \sqrt{q_1} < \sqrt{a}$. Thus, the condition equivalent to the inequality $\kappa \geq 0$ is as follows:

$$\sqrt{q_0} + \sqrt{q_1} \geq \sqrt{a}. \tag{6.13}$$

Note that the inequality $\sqrt{q_0} + \sqrt{q_1} < \sqrt{a}$ yields (6.12). Hence the non-negativity condition can be written in the following form:

$$\forall \bar{x}_0, \bar{x}_1: \ |\bar{x}_0| = |\bar{x}_1| = 1, \ (\bar{x}_0, \bar{x}_1) = 0, \ (Q\bar{x}_0, \bar{x}_1) = 0. \tag{6.14}$$

This condition was not known before. For the two-dimensional case it was discovered and proved by Dmitruk [2]. Condition (6.14) can be regarded as the requirement that Dmitruk’s condition hold for all planes.

Indeed, consider the trajectories lying in the plane $\Gamma$. Then the form $\Phi$ can be written as follows:

$$\Phi = u^2 + 2(PAPx, u) + (PQPx, x),$$

where $P$ is the orthogonal projection onto $\Gamma$. Then $q_0$ and $q_1$ are eigenvectors of the operator $PQP$, and the quantity $\sqrt{a}/2$ is the modulus of the coefficient of the rotation by $\pi/2$ corresponding to the operator $PAP$. Then (6.14) coincides precisely with Dmitruk’s condition for the trajectories in the plane $\Gamma$. Dmitruk’s condition can also be obtained from the frequency criterion, but it does not follow from this criterion in a natural way.
§ 7. A class of quadratic forms reducible to forms with constant coefficients

Consider a pair $\Phi$, $V$ of the following type:

$$
\Phi = a^2(t)u^2 + 2m(t)(Ax,u) + q^2(t)(Qx,x),
$$

$$
V: \quad \dot{x} = k(t)Kx + b(t)Bu.
$$

The corresponding quadratic form is $J = \int_0^\infty \Phi(x(t),u(t),t)\,dt$ and it is considered on $L(V)$. The matrices $A$, $Q$, $K$, $B$ are constant, and the coefficients $a(t)$, $m(t)$, $q(t)$, $k(t)$, $b(t)$ are positive on the half-axis. We assume that the following relations hold:

$$
m = aq; \tag{7.21}
$$

$$
\frac{q^2}{k} = \frac{m}{b}; \tag{7.22}
$$

$$
\frac{d}{dt} \left( \frac{m}{b} \right) = \lambda_0 K \frac{m}{b}, \quad \lambda_0 = \text{const}; \tag{7.23}
$$

$$
\int_0^\infty k(t)\,dt = +\infty. \tag{7.24}
$$

7.1. In this section, under assumptions (7.21)–(7.24), we show that there exists a change of variables reducing the pair $\Phi$, $V$ and the form $J$ to a pair and a form with constant coefficients. Let $\tau(t) = \int_0^t k(t)\,dt$. We shall prove that

$$
e^{-\lambda_0 \tau(t)} = \frac{m(0)}{b(0)} \cdot \frac{b(t)}{m(t)}. \tag{7.3}
$$

Indeed, it follows from (7.21)–(7.23) that

$$
\frac{d}{dt} \left( \frac{m}{b} e^{-\lambda_0 \tau} \right) = \left( \lambda_0 k \frac{m}{b} - \lambda_0 k \frac{m}{b} \right) e^{-\lambda_0 \tau} = 0,
$$

which yields (7.3).

Consider now the change of variables $(x,u,t) \mapsto (\bar{x},\bar{u},\tau)$, where

$$
\tau = \tau(t),
$$

$$
x = e^{-\lambda_0 \tau} \bar{x},
$$

$$
u = \frac{q}{a} e^{-\lambda_0 \tau} \bar{u}.
$$

Then

$$
\Phi(x,u,t) = q^2(t) e^{-\lambda_0 \tau(t)} (\bar{u}^2 + 2A\bar{x}\bar{u} + Q\bar{x}\bar{x}).
$$

Setting $\bar{\Phi}(\bar{x},\bar{u}) = \bar{u}^2 + 2A\bar{x}\bar{u} + Q\bar{x}\bar{x}$ we obtain

$$
\Phi(x,u,t) = q^2(t) e^{-\lambda_0 \tau(t)} \bar{\Phi}(\bar{x},\bar{u}). \tag{7.5}
$$
By (7.3) this means that
\[ \Phi(x, u, t) \, dt = \frac{m(0)}{b(0)} \Phi(x, u) \, d\tau. \]  
(7.6)

Further, replacing \( x, u, t \) in \( V \) by their expressions in terms of \( x, u, \) and \( \tau \) we obtain
\[ k \frac{d}{d\tau} e^{-\frac{\lambda_0}{2} \tau} x = k e^{-\frac{\lambda_0}{2} \tau} Kx + \frac{bq}{a} e^{-\frac{\lambda_0}{2} \tau} Bu. \]

Cancelling out the exponential and \( k \) we obtain
\[ \frac{dx}{d\tau} = \left( \frac{\lambda_0}{2} I + K \right) x + \frac{bq}{ka} B\bar{u}. \]

However, it follows from (7.21) and (7.22) that \( \frac{bq}{ka} = 1. \) Thus, the control system \( V \) is transformed into the control system \( \bar{V}: \)
\[ \frac{d\bar{x}}{d\tau} = \left( \frac{\lambda_0}{2} I + K \right) \bar{x} + B\bar{u}. \]

This procedure transforms the form \( J, \) up to a constant factor, into the form \( \bar{J} = \int_0^\infty \Phi \, d\tau. \) We have thus proved that if the coefficients \( a, m, q, k, b \) satisfy (7.21)–(7.24), then the change (7.4) transforms the pair \( \Phi, V \) into the pair \( \bar{\Phi}, \bar{V} \) with constant coefficients, and the form \( J \) into the form \( \bar{J}. \) Obviously, \( L(V) \) is transformed into \( L(\bar{V}). \) Therefore, any non-negativity condition for \( \bar{J} \) on \( L(\bar{V}) \) is a non-negativity condition for \( J \) on \( L(V). \)

7.2. Let \( \Phi, V \) be a pair of the form (7.1). Setting \( v = b(t)u \) we obtain a new pair \( \Phi_1, V_1, \) where \( \Phi_1 \) and the right-hand side of \( V_1 \) depend on \( x, v, t \) as in (7.1), with coefficients
\[ \frac{a^2}{b^2}, \frac{m}{b}, q^2, k, 1. \]

One can easily verify that if the initial coefficients \( a^2, m, q^2, k, b \) satisfy conditions (7.21)–(7.24), then the new coefficients also satisfy these conditions. Hence we may content ourselves with the case \( b = 1. \) We assume this equality to hold in what follows.

Let \( \Phi, V \) be a pair of the form
\[ \Phi = z(t) \left( \frac{1}{p(t)} u^2 + 2Axu + \rho(t)Qxx \right), \]
\[ V: \ x = \rho Kx + Bu, \]
(7.8)

where \( \rho(t), z(t) > 0; \frac{dz}{dt} = \lambda \rho z; \lambda = \text{const}; \int_0^\infty \rho \, dt = +\infty. \)

We prove the following two propositions.
Proposition 3. Let $\Phi$, $V$ be a pair of the form (7.8). Then the coefficients $a$, $m$, $q$, $k$, $b$, where

$$a = \sqrt{\frac{z}{\rho}}, \quad m = z, \quad q = \sqrt{zp}, \quad k = \rho, \quad b = 1,$$

satisfy conditions (7.21)-(7.24), and $\lambda_0 = \lambda$.

The proof is a direct verification of these conditions.

Proposition 4. Let $\Phi$, $V$ be a pair of the form (7.1) such that $a$, $m$, $q$, $k$, $b$ satisfy (7.21)-(7.24), with $b = 1$. Then the pair $\Phi$, $V$ can be represented in the form (7.8).

Proof. We set $z(t) = m(t)$ and $p(t) = k(t)$. Then

$$\Phi(x, u, t) = z(t)\left(\frac{a^2}{m}u^2 + 2Axu + \frac{q^2}{m}Qxx\right).$$

Since $b = 1$, relations (7.21)-(7.23) yield

$$\frac{a^2}{m} = \frac{1}{k}, \quad \frac{q^2}{m} = k, \quad \frac{dm}{dt} = \lambda_0 km.$$

This proves the proposition.

The representation (7.8) is quite simple. Instead of the four coefficients $a$, $m$, $q$, $k$ ($b = 1$!) it involves just two: $z$ and $p$. If $p = 1$, then the class (7.8) includes the pair

$$\Phi = e^{\lambda t}(u^2 + 2Axu + Qxx),
V: \quad \dot{x} = Kx + Bu$$

for an arbitrary $\lambda$. In accordance with § 7.1, any such pair can be reduced to a pair $\overline{\Phi}$, $\overline{V}$ with constant coefficients, where

$$\overline{\Phi} = \overline{u}^2 + 2A\overline{u}\overline{u} + Q\overline{u}\overline{u},
\overline{V}: \quad \frac{d\overline{u}}{d\tau} = \left(\frac{\lambda}{2} I + K\right)\overline{u} + B\overline{u}.$$

7.3. In this section we consider the form

$$\int_{0}^{1} \left(\theta^\alpha u^2 + 2\theta^\beta (Ax, u) + \theta^\gamma (Qx, x)\right) d\theta,$$

(7.9)

where

$$\frac{dx}{d\theta} = \theta^\mu Kx + Bu, \quad x(1) = 0$$

(7.10)

with $x(\theta) = 0$ and $u(\theta) = 0$ in a neighbourhood of the origin. Such forms were studied by Arutyunov [3] in connection with the failure of the strengthened Legendre condition.
We shall find conditions on \( \alpha, \beta, \gamma, \) and \( \mu \) ensuring that the form (7.9) and the control system (7.10) can be reduced to a form and a control system with constant coefficients on the half-axis. We set \( \kappa(t) = 1 + t \) and consider the change of variables \( \theta = 1/\kappa \). Then the integrand in (7.9) takes the following form:

\[
\Phi(x, u, t) \, dt = \left( \frac{1}{\kappa^{\alpha+2}} u^2 + 2 \frac{1}{\kappa^{\beta+2}} A x u + \frac{1}{\kappa^{\gamma+2}} Q x x \right) \, dt,
\]

and the control system (7.10) is as follows:

\[
V: \quad \frac{d}{dt} x = -\frac{1}{\kappa^{\mu+2}} K x - \frac{1}{\kappa^2} B u.
\] (7.11)

Let us find out when the coefficients of this system satisfy conditions (7.21)-(7.24). Setting \( v = u/\kappa^2 \) we pass to the following pair \( \Phi_1, V_1 \):

\[
\Phi_1(x, v, t) = \kappa^{-\beta} (\kappa^{\beta-\alpha+2} v^2 + 2 A x v + \kappa^{\beta-\sigma-2} Q x x),
\]

\[
V_1: \quad \frac{d}{dt} x = -\frac{1}{\kappa^{\mu+2}} K x - B v.
\] (7.12)

Propositions 3 and 4 show that

\[
\alpha + \gamma = 2\beta, \quad \beta - \gamma = -\mu, \quad \mu + 2 \leq 1.
\]

If \( \beta \neq 0 \), then \( \frac{d}{dt}(\kappa^{-\beta}) = -\beta \kappa^{-\beta-1} \), and therefore

\[
\lambda = -\beta, \quad \rho = \kappa^{-1}, \quad \mu = -1.
\]

If \( \beta = 0 \), then

\[
\lambda = 0, \quad \gamma = -\alpha = \mu, \quad \mu \leq -1, \quad \rho = \kappa^{-\mu-2}, \quad z = 1,
\]

the pair \( \Phi_1, V_1 \) takes the following form:

\[
\Phi_1 = \bar{v}^2 + 2 A \bar{x} \bar{v} + Q \bar{x} \bar{x},
\]

\[
V_1: \quad \frac{d}{d\tau} x = \left( -\frac{\beta}{2} I - K \right) x - B \bar{v},
\]

and the quadratic form \( \bar{J} \) becomes as follows:

\[
\bar{J} = \int_0^\infty \Phi_1(\bar{x}, \bar{v}) \, d\tau.
\]
Bibliography


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