THE NOTION OF RIGIDITY AND OPTIMAL CONTROL THEORY

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ABSTRACT This paper is devoted to the theory of rigid trajectories which are specified by a distribution defined on a manifold. The questions related to rigidity are extensively discussed in the literature. Recently there appeared papers which treated the description problem of rigid trajectories from the point of view of the optimal control theory. In this context we use here the theory of quadratic conditions developed for singular controls. We obtain both necessary and sufficient conditions for the case when the dimension of the distribution is greater than two, which is a new result. A new type of rigidity is studied, which turns out to be connected with sufficient conditions for the local optimality of an abnormal geodesic.

INTRODUCTION

We study the differential inclusion

\[(1.1) \quad \dot{x} \in \Gamma(x), \]

where \(x \in \mathbb{R}^n\) and \(\Gamma(x)\) is an \(m\)-dimensional subspace of \(\mathbb{R}^n\). We assume that the dependence of \(\Gamma\) on \(x\) is smooth—more precisely, that in \(\Gamma(x)\) one can choose a basis \(r_0(x), r_1(x), \ldots, r_{m-1}(x)\) such that the functions \(r_i(x), i = 0, \ldots, m - 1,\) are twice continuously differentiable in \(x\) in the domain of (1.1). We assume for simplicity that this domain is the entire space \(\mathbb{R}^n\). By a trajectory of the differential inclusion (1.1) we will mean a function \(x(t)\) defined on an interval \([t_0, t_1]\), which is Lipschitz continuous on this interval and satisfies (1.1) almost everywhere on \([t_0, t_1]\).

As soon as the basis \(\{r_i(x)\}\) is chosen, (1.1) can be written as a control system

\[(1.2) \quad \dot{x} = \sum_{i=0}^{m-1} u_i r_i(x). \]

Here \(u_i, i = 0, \ldots, m - 1,\) form the control and \(x\) is the state variable.

A trajectory of the system (1.2) is a pair of functions \(x(t), u(t)\) defined on an interval \([t_0, t_1]\), with \(x(t)\) being Lipschitz continuous and \(u(t)\) a measurable function on this interval, which satisfy the system almost everywhere on \([t_0, t_1]\).

Obviously, there is a one-to-one correspondence between the trajectories of the differential inclusion (1.1) and those of the system (1.2). Hence we will deal with the control system (1.2) rather than with the differential inclusion (1.1).

Whether a trajectory of (1.1) is rigid does not depend on the choice of the coordinate system in \(\mathbb{R}^n\). Accordingly, all results to be obtained for the system (1.2) are independent of the choice of the coordinate system in \(\mathbb{R}^n,\) and also of the choice of the functions that form the basis of \(\Gamma(x).\)

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It is a common feature of all conditions for an extremum that they do not depend on the choice of the variables. If they hold, they hold for any choice of variables; if they do not hold, they do not hold for any choice of variables. In our considerations we will frequently use this property, choosing at times the most convenient variables.

We will study the rigidity property of a trajectory of the differential inclusion (1.1). The definition of a rigid trajectory of (1.1) is as follows.

A continuously differentiable trajectory \( x^0(t) \mid [t_0, t_1] \) of the differential inclusion (1.1) is called rigid if any other trajectory \( x(t) \mid [t_0, t_1] \) of the inclusion (1.1) with 
\[
\text{esssup}_{[t_0,t_1]} |\dot{x}(t) - \dot{x}^0(t)| \text{ sufficiently small and } x(t_0) = x^0(t_0), \ x(t_1) = x^0(t_1),
\]
differs from \( x^0(t) \) only by a change of the independent variable \( t \). In other words, there exists a Lipschitz continuous monotone function \( \varphi(t) \) such that \( \varphi(t_0) = t_0, \ \varphi(t_1) = t_1, \) and \( x(t) = x^0(\varphi(t)) \).

However, in the sequel we will assume that \( x^0(t) \) has a continuous third derivative. We will need this assumption to establish both necessary and sufficient quadratic conditions for rigidity.

Moreover, we will assume that \( \dot{x}^0(t) \neq 0 \) for all \( t \in [t_0, t_1] \) and that the function \( x^0(t) \) determines a one-to-one mapping of \([t_0, t_1]\) into \( \mathbb{R}^n \). The former requirement is necessary because a stopping always causes the loss of the rigidity property. The latter is imposed only for simplicity of presentation; having found the rigidity conditions under this assumption, we can easily find them when it fails.

Thus, we assume that

\[
\begin{align*}
\text{the trajectory } x^0(t) \mid [t_0, t_1] & \text{ is thrice continuously differentiable,} \\
(1.3)\quad \dot{x}^0(t) & \neq 0 \quad \forall t \in [t_0, t_1], \\
t', t'' \in [t_0, t_1], \ t' \neq t'' & \implies x^0(t') \neq x^0(t'').
\end{align*}
\]

We will reduce the notion of rigidity to the notion of extremum and derive various conditions for rigidity from conditions for an extremum. We will also consider a number of examples which appear to be interesting. We will see that there exists a stronger notion of rigidity than the one defined above, which has not been identified so far.

2. Reduction of the Notion of Rigidity to the Notion of Extremum

Let \( x^0(t) \mid [t_0, t_1] \) be a trajectory of the differential inclusion (1.1) with properties (1.3). We will construct a modification of the control system (1.2) corresponding to this trajectory.

Let
\[
x^0([t_0, t_1]) = \{x \mid x = x^0(t), \ t \in [t_0, t_1]\}.
\]
Denote by \( t^0(x) \) the inverse mapping to \( t \mapsto x^0(t) \) defined on \( x^0([t_0, t_1]) \).

Let \( \bar{r}_0(x) \) be a twice differentiable vector field on \( \mathbb{R}^n \) such that

\[
\begin{align*}
\bar{r}_0(x) & \neq 0 \quad \forall x \in \mathbb{R}^n, \\
\bar{r}_0(x) & \in \Gamma(x) \quad \forall x \in \mathbb{R}^n, \\
\bar{r}_0(x) & = \dot{x}^0(t^0(x)) \quad \forall x \in x^0([t_0, t_1]).
\end{align*}
\]

The existence of a vector field with these properties is guaranteed by the assumptions that there is a twice differentiable basis in \( \Gamma(x) \) and that \( \Gamma(x) \) is defined on all of \( \mathbb{R}^n \), and by the properties (1.3) of the trajectory \( x^0(t) \mid [t_0, t_1] \). Let \( \tilde{r}_0(x) \) be complemented to a basis \( \tilde{r}_0(x), \ldots, \tilde{r}_{m-1}(x) \) of \( \Gamma(x) \). As we said in Section 1, the differential inclusion
(1.1) can be written as a control system

\[(2.2) \quad \dot{x} = u_0 \hat{r}_0(x) + \sum_{i=1}^{m-1} u_i \hat{r}_i(x).\]

Denote \(u = (u_1, \ldots, u_{m-1})\). Thus the control in the system (2.2) is given by the pair \(u_0(t), u(t)\).

Along with (2.2), consider the control system

\[(2.3) \quad \begin{cases} \frac{dx}{dt} = v \hat{r}_0(x) + \sum_{i=1}^{m-1} u_i \hat{r}_i(x), \\ \frac{dv}{dt} = 0. \end{cases}\]

We will denote trajectories of the system (2.2) by \(\sigma\) and those of (2.3) by \(\varsigma\). We consider trajectories defined on \([t_0, t_1]\). With any trajectory of (2.3) one can associate a trajectory of (2.2) by putting \(u_0(t) = v\) and leaving the other components unchanged. Denote this mapping by \(F\). Obviously, \(F\) is injective.

Let

\[(2.4) \quad \sigma^1 = (x_1(t), u_0^1(t), u^1(t)), \quad \sigma^2 = (x_2(t), u_0^2(t), u^2(t))\]

be trajectories of the system (2.2) defined on \([t_0, t_1]\). We will say that \(\sigma^1\) and \(\sigma^2\) are equivalent and write \(\sigma^1 \sim \sigma^2\), if there exists a monotone increasing Lipschitz continuous function \(\varphi\) defined on \([t_0, t_1]\) such that \(\varphi(t_0) = t_0, \varphi(t_1) = t_1\), and

\[\varphi(t) \geq \text{const} > 0, \quad t \in [t_0, t_1],\]

satisfying the condition

\[x_2(t) = x_1(\varphi(t)).\]

This implies that

\[(2.5) \quad u_0^2(t) = \varphi(t) u_0^1(\varphi(t)), \quad u^2(t) = \varphi(t) u^1(\varphi(t)).\]

Set \(\sigma = (x(t), u_0(t), u(t) \mid [t_0, t_1])\) and let \(u_0(t) \geq \text{const} > 0\).

**Proposition 2.1.** There exists a unique trajectory \(\varsigma\) of the system (2.3) defined on \([t_0, t_1]\) such that \(\sigma \sim F(\varsigma)\).

**Proof.** We will look for \(\varphi\) specified by the conditions

\[(2.6) \quad \varphi(t_0) = t_0, \quad \varphi(t_1) = t_1, \quad \frac{d\varphi}{dt} u^0(\varphi(t)) = v = \text{const}.\]

It is easily seen that \(\varphi\) is uniquely defined by these conditions and

\[v = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} u^0(t) \, dt.\]

Let \(\varsigma = (x(\varphi(t)), v, \frac{d\varphi}{dt} u(\varphi(t)))\). Then \(\varsigma\) is a trajectory of the system (2.3) defined on \([t_0, t_1]\), and (2.6) implies that \(\sigma \sim F(\varsigma)\). Its uniqueness follows from uniqueness of the solution of (2.6) with the specified boundary conditions.
Consider the following problem on the interval \([t_0, t_1]\) for the system (2.3):

\[
J = \int_{t_0}^{t_1} y^2(t) \, dt \to \text{max}, \quad y(t_0) = 0,
\]

\[J_t \quad (2.7)
\]

\[
x(t_0) = x^0(t_0), \quad x(t_1) = x^0(t_1),
\]

\[
\frac{dy}{dt} = u(t), \quad \frac{dx}{dt} = v\hat{r}_0(x) + \sum_{i=1}^{m-1} u_i \hat{r}_i(x), \quad \frac{dv}{dt} = 0,
\]

where \(y = (y_1, \ldots, y_{m-1}) \in \mathbb{R}^{m-1}\). Let \(x^0 = (x^0(t), v^0, u^0(t))\), where \(v^0 = 1, u^0(t) = 0\). By the choice of \(\hat{r}_0(x)\), \(x^0\) is a trajectory of the system (2.3) defined on \([t_0, t_1]\).

**Theorem 2.1.** The trajectory \(x^0(t) \mid [t_0, t_1]\) is a rigid solution of the differential inclusion (1.1) if and only if \(x^0\) is a weak maximum in the problem (2.7).

**Proof.** Assume that \(x^0\) is a weak maximum in the problem (2.7). This means the following. Let \(x^s, s = 1, 2, \ldots\), be a sequence of trajectories of the system (2.3) defined on \([t_0, t_1]\) and such that

\[
x^s(t_0) = x^0(t_0), \quad x^s(t_1) = x^0(t_1), \quad s = 1, 2, \ldots,
\]

\[
v^s \to v^0, \quad \text{ess sup}_{[t_0, t_1]} |u^s(t) - u^0(t)| \to 0, \quad \text{as} \quad s \to \infty.
\]

Then one can find \(s_0 \geq 1\) such that

\[
J(x^s) \leq J(x^0) \quad \forall s \geq s_0,
\]

where \(J(x^s) = \int_{t_0}^{t_1} (y^s(t))^2 \, dt, s = 0, 1, \ldots\).

Since \(u^0(t) = 0 \mid [t_0, t_1]\), we have \(J(x^0) = 0\). On the other hand, \(J(x) \geq 0\) for all \(x\). Then (2.9) implies that \(J(x) = 0\) for all trajectories \(x\) of the system (2.3) which are sufficiently close to \(x^0\) in the weak sense, i.e., in the sense of the norm which determines the notion of a weak minimum.

Now, let \(x^s(t), s = 1, 2, \ldots\), be a sequence of solutions of (1.1) weakly converging to \(x^0(t)\) and satisfying the same boundary conditions. In other words,

\[
x^s(t_0) = x^0(t_0), \quad x^s(t_1) = x^0(t_1), \quad s = 1, 2, \ldots,
\]

\[
\text{ess sup}_{[t_0, t_1]} |x^s(t) - x^0(t)| \to 0, \quad s \to \infty.
\]

In terms of the system (2.2), this yields a sequence of its trajectories \(\{\sigma^s\}\) converging to \(\sigma^0\), i.e.,

\[
\text{ess sup}_{[t_0, t_1]} |u^s_0(t) - u^0_0(t)| \to 0,
\]

\[
\text{ess sup}_{[t_0, t_1]} |u^s(t) - u^0(t)| \to 0, \quad s \to \infty.
\]

Since \(u^0_0(t) = 1, u^0(t) = 0 \mid [t_0, t_1]\), we have

\[
\text{ess sup}_{[t_0, t_1]} |u^s_0(t) - 1| \to 0, \quad \text{ess sup}_{[t_0, t_1]} |u^s(t)| \to 0, \quad s \to \infty.
\]

Therefore we can apply Proposition 2.1 to the sequence \(\{\sigma^s\}\).

Let \(x^s\) be a trajectory of the system (2.3) such that \(\sigma^s \sim F(x^s)\). It follows from (2.6) and (2.11) that

\[
\text{ess sup}_{[t_0, t_1]} |\dot{x}^s - 1| \to 0, \quad s \to \infty.
\]
Therefore \( x^s \rightarrow x^0 \) in the weak sense. This implies that \( J(x^s) = 0 \) for \( s \) greater than some \( s_0 \). In its turn, this implies that \( u^s(t) = 0 \mid [t_0, t_1] \) for \( s \geq s_0 \).

Thus \( x^s(t) \) for \( s \geq s_0 \) satisfies the equation \( \dot{x}^s = u^s_0(t)\hat{r}_0(x^s) \). This implies that for \( s \geq s_0 \) the solutions \( x^s(t) \) and \( x^0(t) \) are related to each other by a change of the independent variable. Then \( x^0(t) \mid [t_0, t_1] \) is a rigid trajectory of (1.1). This proves the first part of the theorem. Let us prove the second part.

Let \( x^0(t) \mid [t_0, t_1] \) be a rigid trajectory of the differential inclusion (1.1). Let \( \{x^s\} \) be a sequence of trajectories weakly converging to \( x^0 \). Since \( \sigma^0 = F(x^0) \), we then have \( \sigma^s \rightarrow \sigma^0 \), where \( \sigma^s = F(x^s) \). Since \( x^0(t) \) is a rigid trajectory, we have \( u^s(t) = 0 \mid [t_0, t_1] \) for \( s \geq s_0 \). Hence \( J(x^s) = 0 \), \( s \geq s_0 \). Thus we have established that \( x^0 \) is a weak minimum in the problem (2.7). This completes the proof of the second part and hence the theorem.

This theorem allows us to derive rigidity conditions from conditions for a maximum. We will primarily use the results by Milyutin and Dmitruk on the theory of quadratic conditions for an extremum in optimal control problems. In the next section, making use of the maximum principle, we will obtain and compare various extremality conditions in the systems (2.2) and (2.3).

3. NECESSARY FIRST ORDER CONDITIONS

Let \( r_i(x) \), \( i = 0, \ldots, m - 1 \), be a basis in \( \Gamma(x) \). Consider the control system

\[
\frac{dx}{dt} = u_0(t) r_0(x) + \sum_{i=1}^{m-1} u_i(t) r_i(x).
\]

According to optimal control theory [1], an extremal of the control system (3.1) is defined as follows. A quadruple of functions \( \psi(t), x(t), u_0(t), u(t) \) defined on an interval \( [t_0, t_1] \) is called an extremal of the system (3.1) if \( x(t), u_0(t), u(t) \mid [t_0, t_1] \) is a trajectory of the system (3.1), \( \psi(t) (\psi \in \mathbb{R}^n) \) is Lipschitz continuous on \( [t_0, t_1] \),

\[
-\frac{d\psi}{dt} = H'(\psi(t), x(t), u_0(t), u(t)),
\]

where

\[
H = u_0 \psi r_0(x) + \sum_{i=1}^{m-1} u_i \psi r_i(x),
\]

and, finally, \( \psi(t) \neq 0 \mid [t_0, t_1] \).

The last condition can be replaced by condition \( \exists t : \psi(t) \neq 0 \), since the equation satisfied by \( \psi \) (the adjoint equation) is linear with respect to \( \psi \).

Under a change of the basis of \( \Gamma(x) \) or the coordinate system in \( \mathbb{R}^n \) the sets of extremals transform into one another by well-known formulas [1]. The form \( \psi dx - H \, dt \) is invariant under these transformations. Thus we can speak of extremals of the differential inclusion (1.1).

Let \( \psi(t), x(t), u_0(t), u(t) \) be an extremal of the system (3.1). Then the trajectory \( (x(t), u_0(t), u(t)) \) is called the trajectory component of the extremal.

Let \( x^0(t) \mid [t_0, t_1] \) be a rigid trajectory of the differential inclusion (1.1). As in Section 2, take vector fields \( \hat{r}_0(x), \ldots, \hat{r}_{m-1}(x) \) which form a basis of \( \Gamma(x) \), and consider the system (2.3).

Consider the trajectory \( x^0 = (x^0(t), y^0 = 1, u^0(t) = 0) \). By Theorem 2.1 \( x^0 \) is a weak minimum in the problem (2.7); hence it satisfies the conditions of the local maximum principle. However in our case the right-hand sides of the system (2.3) depend on the
control linearly, and then the local maximum principle coincides with Pontryagin’s maximum principle, which consists in the following.

The Pontryagin function in the problem (2.7) has the form

\[ \hat{H} = \alpha_0 y^2 + \psi_y u + v\hat{r}_0(x) + \sum_{i=1}^{m-1} u_i \psi \hat{r}_i(x). \]

According to the maximum principle, there exist a constant \( \alpha_0 \geq 0 \) and functions \( \psi_y(t) \) \((\psi_y \in \mathbb{R}^{m-1}), \psi(t) \) \((\psi \in \mathbb{R}^n)\), and \( \psi_v \) \((\psi_v \in \mathbb{R}^1)\) such that

\[
\begin{align*}
\psi_y(t_1) &= 0, \quad \psi_v(t_0) = \psi_v(t_1) = 0, \\
-\dot{\psi}_y &= \hat{H}'_y \left( \alpha_0, \psi_y(t), \psi(t), \psi_v(t); \ y^0(t), x^0(t), v^0, u^0(t) \right), \\
-\dot{\psi} &= \hat{H}' \left( \alpha_0, \psi_y(t), \psi(t), \psi_v(t); \ y^0(t), x^0(t), v^0, u^0(t) \right), \\
-\dot{\psi}_v &= \hat{H}'_v \left( \alpha_0, \psi_y(t), \psi(t), \psi_v(t); \ y^0(t), x^0(t), v^0, u^0(t) \right), \\
\psi_y(t) + \psi(t)\hat{r}_0(x^0(t)) &= 0 \quad \text{for } i = 1, \ldots, m-1, \\
\alpha_0^2 + \int_{t_0}^{t_1} (\psi_y^2(t) + \psi^2(t) + \psi_v^2(t)) \, dt &> 0.
\end{align*}
\]

For the trajectory \( x^0 \) these conditions can be significantly simplified. Since \( y^0(t) = 0 \) \([t_0, t_1], v^0 = 1\), and \( u^0(t) = 0 \) \([t_0, t_1]\), it follows that the adjoint equations (3.4) take the form

\[
\begin{align*}
-\dot{\psi}_y &= 0; \\
-\dot{\psi} &= \psi \hat{r}'_{xx} (x^0(t)); \\
-\dot{\psi}_v &= \psi \hat{r}_0 (x^0(t)).
\end{align*}
\]

Then we obtain from the transversality conditions (3.3) that \( \psi_y = 0 \). Since the problem (2.7) is autonomous, i.e., the right-hand sides of the system (2.3) and the integrand do not contain explicitly the variable \( t \), we have

\[
\hat{H} \left( \alpha_0, \psi_y(t), \psi(t), \psi_v(t); y^0(t), x^0(t), v^0, u^0(t) \right) = \text{const}.
\]

But \( \hat{H} \left( \alpha_0, \psi_y(t), \psi(t), \psi_v(t); y^0(t), x^0(t), v^0, u^0(t) \right) = \psi(t)\hat{r}_0(x^0(t)) \). Then \(-\dot{\psi}_v = \text{const}; hence, in view of (3.3), we obtain \( \psi_v(t) = 0 \) \([t_0, t_1]\) and, therefore,

\[
\psi(t)\hat{r}_0(x^0(t)) = 0.
\]

It follows from (3.7) and (3.8) that the collection \( \psi(t), x^0(t), u^0(t), u^0(t) \) \([t_0, t_1]\) satisfies all the extremality conditions in the system (2.2) except for the last one. Here \( u^0(t) = 1 \) \([t_0, t_1]\), \( u^0(t) = 0 \) \([t_0, t_1]\).

Put \( \lambda = (\alpha_0, \psi(t)) \) and introduce the set

\[
\Lambda_0 = \left\{ \lambda \mid \alpha_0 \geq 0, \ \psi(t) \text{ such that (3.3), (3.7), and (3.8); } \alpha_0^2 + \int_{t_0}^{t_1} \psi^2(t) \, dt = 1 \right\}.
\]

The maximum principle is the property that \( \Lambda_0 \neq \emptyset \).

Let \( \Psi_0 \) be the set of all \( \psi(t) \) \([t_0, t_1]\) such that \( \psi(t), \sigma^0 \) is an extremal of (2.2) and

\[
\int_{t_0}^{t_1} \psi^2(t) \, dt = 1.
\]

Then it is easily verified that

\[
\lambda \in \Lambda_0 \iff \psi(t) = \beta \hat{\psi}(t) \mid \hat{\psi}(t) \in \Psi_0, \quad \beta \geq 0, \quad \alpha_0^2 + \beta^2 = 1.
\]
If $\Psi_0 = \emptyset$, then $\beta = 0$ and, by the definition of $\beta$, $\hat{\psi}(t) = 0$. Thus the structure of $\Lambda_0$ is closely related to certain extremals of the system (2.2). The set $\Lambda_0$ contains an element $\lambda_*$ for which $\psi(t) = 0 \mid [t_0, t_1]$ and $\alpha_0 = 1$.

**Proposition 3.1.**

\[(3.11) \quad \Lambda_0 \setminus \{\lambda_*\} \neq \emptyset.\]

*Proof.* Assume the contrary, i.e., that $\Lambda_0$ consists of a single element $\lambda = \lambda_*$. Denote $w = (y, x, v, u)$. Then $w^0(t) = (y^0(t), x^0(t), v^0(t), u^0(t))$. Put

$$\tilde{H}(\alpha_0, \psi, w) = \alpha_0 y^2 + H(\psi, x, v, u),$$

where $H$ is the Pontryagin function of the system (2.3). Obviously,

$$\tilde{H}(\alpha_0, \psi, w) = \tilde{H}(\alpha_0, \psi_y = 0, \psi, \psi_v = 0, w).$$

For a given $\lambda = (\alpha_0, \psi(t))$ consider the quadratic form

\[(3.12) \quad \omega (\lambda, \bar{w}(\cdot)) = -\frac{1}{2} \int_{t_0}^{t_1} \tilde{H}_w w^0(t) \bar{w}(t) \bar{w}(t) dt,
\]

where $\bar{w} = (\bar{y}, \bar{x}, \bar{v}, \bar{u})$. When $\lambda = \lambda_*$,

\[(3.13) \quad \omega (\lambda_*, \bar{w}(\cdot)) = - \int_{t_0}^{t_1} \bar{y}^2(t) dt.
\]

Now we introduce the notion of a critical variation, which plays an important role in the theory of quadratic conditions. The variation $\bar{w}(t)$ is called *critical* if it satisfies the system of equations

\[(3.14) \quad \dot{\bar{y}} = \bar{u},
\]

\[\dot{\bar{x}} = \bar{r}_0 \bar{x} + \bar{v} \bar{r}_0(x^0(t)) + \sum_{i=1}^{m-1} \bar{u}_i \bar{r}_i(x^0(t)),
\]

\[\dot{\bar{v}} = 0,
\]

and the conditions

\[(3.15) \quad \int_{t_0}^{t_1} \bar{y}(t) \bar{y}(t) dt \geq 0, \quad \bar{x}(t_0) = \bar{x}(t_1) = 0, \quad \bar{y}(t_0) = 0.
\]

The set of critical variations forms the cone of critical variations, denoted by $\mathcal{K}$.

The first condition (3.15) is always fulfilled, since $y^0(t) = 0 \mid [t_0, t_1]$. Thus conditions (3.15) impose only boundary conditions.

It is seen from (3.14) that $\bar{y}(t), \bar{x}(t)$ are uniquely and linearly determined by $\bar{v}, \bar{u}(t)$.

Conditions (3.15) imply that $\mathcal{K}$ is a nonempty infinite-dimensional linear manifold in the space of pairs $\bar{v}, \bar{u}(t)$.

We know from the theory of quadratic conditions [2] that the inequality

\[(3.16) \quad \max_{\Lambda_0} \omega (\lambda; \bar{w}(\cdot)) \geq 0 \quad \forall \bar{w}(\cdot) \in \mathcal{K}
\]

is a necessary condition for a weak minimum in the problem (2.7).

Since $\Lambda_0 = \{\lambda_*\}$, taking (3.13) into account we infer from this condition that

$$- \int_{t_0}^{t_1} \bar{y}^2(t) dt \geq 0 \quad \forall \bar{w}(t) \in \mathcal{K}.$$
However, this inequality implies that \( \bar{y}(t) = 0 \mid [t_0, t_1] \), and therefore, that \( \bar{u}(t) = 0 \mid [t_0, t_1] \). But this contradicts the fact that the dimension of \( \mathcal{K} \) is infinite. Thus the proposition is proved. \( \square \)

This proposition implies that \( \Psi_0 \neq \emptyset \). Thus we have established that whenever \( x^0(t) \mid [t_0, t_1] \) is a rigid trajectory of the inclusion (1.1), \( x^0(t) \) is the state component of an extremal of the system (2.2). But extremals are transformed into one another by well-known rules according to the choice of coordinates in \( \mathbb{R}^n \) and basis of \( \Gamma(x) \). In particular, the adjoint \( (\psi(t)) \) and state \( (x(t)) \) components of an extremal do not depend on the choice of a basis of \( \Gamma(x) \). Therefore we can speak of extremals of the differential inclusion (1.1). Thus we have obtained the following result.

**Theorem 3.1.** If \( x^0(t) \) is a rigid trajectory of the differential inclusion (1.1) satisfying conditions (1.3), then it is a state component of an extremal of (1.1).

Since the adjoint and state components do not depend on the choice of the basis, in order to find rigid extremals one can start, according to this theorem, with determination of extremals of the control system for an arbitrary basis and then, after the trajectories have been found, turn to the system (2.2). This system is also not unique, but the conditions for rigidity to be obtained will not depend on its choice.

In the next section we will see that not every extremal is connected with rigid trajectories, and this must be taken into account when looking for rigid trajectories of the differential inclusion (1.1).

### 4. Necessary Quadratic Conditions

Let, as before, \( x^0(t) \mid [t_0, t_1] \) be a rigid trajectory of the differential inclusion (1.1), satisfying conditions (1.3). For the trajectory \( x^0(t) \) we choose a basis \( r_0(x), \ldots, r_{m-1}(x) \) satisfying the requirements (2.1), and consider the trajectory \( x^0 \) of the control system (2.3). By Theorem 2.1 the trajectory \( x^0 \) affords a weak maximum in the problem (2.7). Hence necessary conditions for an extremum obtained in optimal control theory are fulfilled.

In this section we will use the necessary quadratic condition for a weak extremum obtained by A. A. Milyutin [3] for problems with a finite-dimensional image, as the problem (2.7) is. This condition, as applied to the trajectory \( x^0 \) in the problem (2.7), is as follows.

We will write \( A \subset A^0 \) if there exists a linear manifold \( \mathcal{L} \subset \mathcal{K} \) of finite codimension in \( \mathcal{K} \) such that

\[
\lambda \in \Lambda_0^+, \quad \omega(\lambda, \bar{w}(\cdot)) \geq 0 \mid \mathcal{L}.
\]

Then the condition

\[
\Lambda_0^+ \neq \emptyset, \quad \sup_{\Lambda_0^+} \omega(\lambda, \bar{w}(\cdot)) \geq 0 \quad \forall \bar{w}(\cdot) \in \mathcal{K}
\]

is necessary for \( x^0 \) to be a weak minimum.

First of all, note that \( \lambda \notin \Lambda_0^+ \), since

\[
\omega(\lambda, \bar{w}(\cdot)) < 0 \quad \forall \bar{w}(\cdot) \quad \text{such that} \quad \bar{y}(t) \neq 0 \mid [t_0, t_1],
\]

and the set of such elements \( \omega(\cdot) \) has codimension 1 in \( \mathcal{K} \).

Let \( \lambda = (\alpha_0, \psi(\cdot)) \in \Lambda_0^+ \). By (3.10), \( \psi(t) = \beta \tilde{\psi}(t) \), where \( \tilde{\psi}(\cdot) \in \Psi_0 \). Let

\[
\bar{w}(\tilde{\psi}(\cdot), \bar{w}(\cdot)) = -\frac{1}{2} \int_{t_0}^{t_1} H''_{\tilde{w}\bar{w}}(\tilde{\psi}(t), \bar{w}(t)) \bar{w}(t) \bar{w}(t) dt.
\]
Then, obviously, (3.12) implies that

\[
\phi(\lambda, \bar{w}(\cdot)) = -\alpha_0 \int_{t_0}^{t_1} \bar{y}^2 \, dt + \beta \bar{w}(\psi(\cdot), \bar{w}(\cdot)) \quad \forall \bar{w}(t) \mid [t_0, t_1].
\]

Since \( \beta = \sqrt{1 - \alpha_0^2} > 0 \), this implies that \( \tilde{w}(\psi(\cdot), \bar{w}(\cdot)) \) is nonnegative on a linear manifold of finite codimension in \( \mathcal{K} \). By definition, such elements \( \Psi_0 \) constitute the set \( \Psi_0^+ \). Thus we have shown that \( \Psi_0^+ \neq \emptyset \).

Let \( \bar{w}(t) \in \mathcal{K} \). Then from condition (4.2), formula (4.3), and the inclusion

\[
\tilde{w}(\cdot) \in \Psi_0^+,
\]

which was obtained above, we obtain

\[
\sup_{\alpha_0} \left( -\alpha_0 \int_{t_0}^{t_1} \bar{y}^2 \, dt + \sqrt{1 - \alpha_0^2} \cdot \sup_{\Psi_0^+} \tilde{w}(\psi(\cdot), \bar{w}(\cdot)) \right) \geq 0.
\]

Hence it is easily seen that if \( \int_{t_0}^{t_1} \bar{y}^2 \, dt > 0 \), then

\[
\sup_{\Psi_0^+} \tilde{w}(\psi(\cdot), \bar{w}(\cdot)) > 0.
\]

Since the set of such elements is dense in \( \mathcal{K} \), the inequality (4.6) holds for any element of \( \mathcal{K} \). Thus

\[
\sup_{\Psi_0^+} \tilde{w}(\psi(\cdot), \bar{w}(\cdot)) \geq 0 \quad \forall \bar{w}(\cdot) \in \mathcal{K}.
\]

In this condition we write \( \psi \) instead of \( \tilde{w} \). This is the required form of the necessary condition. It involves only characteristics of the control system (2.3), but no information on the problem (2.7). One can show that this condition does not depend on the choice of coordinates in \( \mathbb{R}^n \), nor on the choice of the basis \( \vec{e}_0(x), \ldots, \vec{e}_{m-1}(x) \) of \( \Gamma(x) \).

We will investigate this condition and derive its consequences. We will need the Goh transformation of a linear control system, which is well known in optimal control theory. It is as follows. Let the control system have the form

\[
z(t) = A(t)z + B(t)u,
\]

where \( A(t) \) and \( B(t) \) are matrix functions of \( t \), \( u \in \mathbb{R}^{m-1} \), and \( z \in \mathbb{R}^{n+1} \). We impose the last two requirements in order that the system (4.8) may fit closer to our case. Let us introduce new variables \( \zeta \) and \( y \), \( \zeta \in \mathbb{R}^{n+1}, y \in \mathbb{R}^{m-1} \).

The Goh transformation has the form

\[
(z(t), u(t)) \mapsto (\zeta(t), y(t)),
\]

where \( \zeta \) and \( y \) are defined by

\[
\dot{y} = u, \quad y(t_0) = 0, \quad \zeta(t) = z(t) - B(t) y(t).
\]

There is the following relationship between \( \zeta(t) \) and \( y(t) \):

\[
\dot{\zeta} = A\zeta + (AB - B)y.
\]

Thus \( y \) in (4.10) is the control and \( \zeta \) is the state variable. This setting played an important role in deriving quadratic conditions of a very fine order \( \gamma(y) = \int_{t_0}^{t_1} y^2(t) \, dt \), which was crucial for obtaining sufficient conditions in optimal control problems that are linear in control and have no local constraints (Dmitruk [4]), and eventually for obtaining sufficient conditions of rigidity.
A A Milyutin

Let us apply the Goh transformation to the system that we obtain by linearization of (2.3) at the point \( x^0 \). It has the form

\[
\dot{x} = \tilde{r}_{0x}(x^0(t)) \bar{x} + \bar{v}\tilde{r}_0(x^0(t)) + \sum_{i=1}^{m-1} \bar{u}_i \tilde{r}_i(x^0(t)),
\]

\[
\dot{\bar{v}} = 0.
\]

The Goh transformation has the form

\[
\begin{align*}
\dot{\bar{y}} &= \bar{u}, & \bar{y}(t_0) &= 0, \\
\bar{\xi} &= \bar{x} - \sum_{i=1}^{m-1} \bar{y}_i \tilde{r}_i(x^0(t)), & \bar{\xi} &\in \mathbb{R}^n,
\end{align*}
\]

leaving \( \bar{v} \) unchanged.

The equation (4.10) becomes

\[
\begin{align*}
\dot{\bar{\xi}} &= \tilde{r}_{0x}(x^0(t))\bar{\xi} + \bar{v}\tilde{r}_0(x^0(t)) + \sum_{i=1}^{m-1} \bar{y}_i [\tilde{r}_0, \tilde{r}_i](x^0(t)), \\
\dot{\bar{v}} &= 0,
\end{align*}
\]

where \([r, \rho](x) = r'_{x}(x)\rho(x) - r_{x}(x)\rho(x)\) is the Lie commutator of vector fields \( r(x) \) and \( \rho(x) \).

By definition the quadratic form \( \bar{\omega} \) can be written as \( \bar{\omega} = \bar{\omega}_1 + \bar{\omega}_2 \), where

\[
\begin{align*}
\bar{\omega}_1(\psi(\cdot), \bar{w}(\cdot)) &= \int_{t_0}^{t_1} \left( \frac{1}{2} \psi(t) \tilde{r}_{0xx}(x^0(t)) \bar{x}(t) \bar{x}(t) + \bar{v} \psi(t) \tilde{r}_{0x}(x^0(t)) \bar{x}(t) \right) dt, \\
\bar{\omega}_2 &= \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \bar{u}_i(t) \psi(t) \tilde{r}_{ix}(x^0(t)) \bar{x}(t) dt.
\end{align*}
\]

Replacing \( \bar{x}(t) \) by its expression through \( \bar{\xi} \) and \( \bar{y} \) according to the Goh transformation, we immediately obtain that the form \( \bar{\omega}_1 \) depends only on these variables.

As for \( \bar{\omega}_2 \), it can also be transformed to a form which does not involve \( \bar{u}(t) \). This fact, however, is a consequence of the necessary condition (4.7). As the first step, we perform the following transformation of the form \( \bar{\omega}_2 \). We have by (4.14)

\[
\bar{\omega}_2 = \bar{\omega}_{21} + \bar{\omega}_{22},
\]

where

\[
\begin{align*}
\bar{\omega}_{21} &= \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \bar{u}_i(t) \psi(t) \tilde{r}_{ix}(x^0(t)) \bar{\xi}(t) dt, \\
\bar{\omega}_{22} &= \int_{t_0}^{t_1} \sum_{i,k=1}^{m-1} \bar{u}_i(t) \bar{y}_k(t) \psi(t) \tilde{r}_{ix}(x^0(t)) \tilde{r}_k(x^0(t)) dt.
\end{align*}
\]
RIGIDITY AND OPTIMAL CONTROL

III. A

\[\begin{align*}
\bar{\omega}_{21} &= \sum_{i=1}^{m-1} \bar{y}_i(t_1) \psi(t_1) \mathcal{F}_{i\mathbf{x}}(x^0(t_1)) \mathcal{\dot{\xi}}(t_1) - \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \bar{y}_i(t) \left( \psi(t) \mathcal{F}_{i\mathbf{x}}(x^0(t)) \right) \mathcal{\dot{\xi}}(t) \, dt \\
&\quad - \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \bar{y}_i(t) \psi(t) \mathcal{\dot{\mathcal{F}}}_{i\mathbf{x}}(x^0(t)) \left( \mathcal{F}_{i\mathbf{x}}(x^0(t)) \mathcal{\dot{\xi}}(t) + \bar{\nu} \mathcal{\dot{\mathcal{F}}}_0(x^0(t)) \right) \, dt \\
&\quad - \int_{t_0}^{t_1} \sum_{i, k=1}^{m-1} \bar{y}_i(t) \bar{y}_k(t) \psi(t) \mathcal{\dot{\mathcal{F}}}_{i\mathbf{x}}(x^0(t)) \left[ \mathcal{F}_0, \mathcal{F}_k \right](x^0(t)) \, dt.
\end{align*}\]

Put

\[f_{ik}(t) = \psi(t) \mathcal{\dot{\mathcal{F}}}_{i\mathbf{x}}(x^0(t)) \mathcal{\dot{\mathcal{F}}}_k(x^0(t)), \quad i, k = 1, \ldots, m - 1.\]

Then

\[\bar{\omega}_{22} = \int_{t_0}^{t_1} \left( \sum_{i, k=1}^{m-1} \left( \bar{u}_i(t) \bar{y}_k(t) f_{ik}(t) + \bar{u}_k(t) \bar{y}_i(t) f_{ki}(t) \right) + \sum_{i=1}^{m-1} \bar{u}(t) \bar{y}_i(t) f_{ii}(t) \right) \, dt.
\]

Transform the terms of the first sum by the formula

\[aA + bB = (a + b) \left( \frac{A + B}{2} \right) + (a - b) \left( \frac{A - B}{2} \right).\]

Rearranging the resulting terms, we can write \(\bar{\omega}_{22}\) as the sum

\[\bar{\omega}_{22} = \bar{\omega}_{221} + \bar{\omega}_{222},\]

where

\[\begin{align*}
\bar{\omega}_{221} &= \int_{t_0}^{t_1} \left( \sum_{i, k=1}^{m-1} \left( \bar{u}_i(t) \bar{y}_k(t) f_{ik}(t) + \bar{u}_k(t) \bar{y}_i(t) f_{ki}(t) \right) + \sum_{i=1}^{m-1} \bar{u}(t) \bar{y}_i(t) f_{ii}(t) \right) \, dt, \\
\bar{\omega}_{222} &= \int_{t_0}^{t_1} \sum_{i, k=1}^{m-1} \left( \bar{u}_i(t) \bar{y}_k(t) - \bar{u}_k(t) \bar{y}_i(t) \right) f_{ik}(t) - f_{ki}(t) \, dt.
\end{align*}\]

Integrating by parts, rewrite \(\bar{\omega}_{221}\) as

\[\begin{align*}
\bar{\omega}_{221} &= \sum_{i, k=1}^{m-1} \bar{y}_i(t_1) \bar{y}_k(t_1) \frac{f_{ik}(t_1) + f_{ki}(t_1)}{2} - \sum_{i=1}^{m-1} \frac{1}{2} \bar{y}_i^2(t_1) f_{ii}(t_1) \\
&\quad - \int_{t_0}^{t_1} \left( \sum_{i, k=1}^{m-1} \bar{y}_i(t) \bar{y}_k(t) \frac{f_{ik}(t) + f_{ki}(t)}{2} + \frac{1}{2} \sum_{i=1}^{m-1} \bar{y}_i^2(t) \dot{f}_{ii}(t) \right) \, dt.
\end{align*}\]

Hence

\[\bar{\omega}_{221} = \frac{1}{2} \sum_{i, k=1}^{m-1} \bar{y}_i(t_1) \bar{y}_k(t_1) f_{ik}(t_1) - \frac{1}{2} \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \bar{y}_i(t) \bar{y}_k(t) \dot{f}_{ik}(t) \, dt.
\]

It is proved in the theory of quadratic conditions [3, 6] that

\[\psi(\cdot) \in \Psi_0^+ \implies \bar{\omega}_{222} = 0 \quad \forall \bar{y}(t).\]
Therefore

\[ \psi(\cdot) \in \Psi_0^+ \implies f_{ik}(t) - f_{ki}(t) = 0 \ | \ [t_0, t_1], \quad i, k = 1, \ldots, m - 1. \]

Since the condition \( \Psi_0^+ \neq \emptyset \) is necessary for rigidity, these equalities are also necessary for rigidity.

But \( f_{ik}(t) - f_{ki}(t) = \psi(\cdot)[\widehat{r}_i, \widehat{r}_k](x^0(t)) \).

Thus the equalities

\[ \psi[\widehat{r}_i, \widehat{r}_k](x^0(t)) = 0 \ | \ [t_0, t_1], \quad i, k = 1, \ldots, m - 1, \]

constitute a necessary condition for rigidity.

Moreover, we can add to them the equalities

\[ \psi[\widehat{r}_0, \widehat{r}_k](x^0(t)) = 0 \ | \ [t_0, t_1], \quad i, k = 0, 1, \ldots, m - 1, \]

which result from differentiation of the equality

\[ \psi(t)[\widehat{r}_0](x^0(t)) = 0 \ | \ [t_0, t_1] \]

with respect to \( t \).

Finally, we obtain:

\[ (4.18) \quad \psi(t)[\widehat{r}_i, \widehat{r}_k](x^0(t)) = 0 \ | \ [t_0, t_1], \quad i, k = 0, 1, \ldots, m - 1. \]

It is seen that this necessary condition determines a set of extremals, which is invariant with respect to the choice of the basis and the coordinate system in \( \mathbb{R}^n \). Therefore it can be used to select candidates for rigidity from among extremals of the differential inclusion (1.1).

Let us describe the situation which appears to us the most appropriate for finding extremals satisfying condition (4.18). Let a differential inclusion (1.1) be given and let a basis \( (r_0(x), \ldots, r_{m-1}(x)) \) in \( \Gamma(x) \) be chosen.

Let \( M_0 \) be the set of pairs \( (\psi, x) \in \mathbb{R}^{2n} \) satisfying the following conditions:

(a) \( \psi[r_i](x) = 0, \quad i = 0, \ldots, m - 1, \)

(b) \( \psi[r_i, r_k](x) = 0, \quad i, k = 0, \ldots, m - 1, \)

(c) \( \text{rank}\{\Theta_{ik}\}_{i,k=0,\ldots,m-1} = m - 1, \)

where

\[ (4.19) \quad \Theta_{ik} \in \mathbb{R}^m, \quad \Theta_{ik} = (\Theta_{ik,0}, \ldots, \Theta_{ik,m-1}), \]

\[ \Theta_{ik,s} = \psi[r_s[r_i, r_k]](x). \]

One can see that the set \( M_0 \) does not depend on the choice of the basis in \( \Gamma(x) \) and is invariant with respect to the choice of the coordinate system in \( \mathbb{R}^n \). We adopt the set \( M_0 \) because on this set the control \( u \) is uniquely defined (up to a constant factor) if we restrict ourselves to extremals satisfying condition (4.18). Indeed, for such an extremal the following conditions must be satisfied:

\[ (4.20) \quad \frac{d}{dt} \psi[r_i, r_k](x) = 0, \quad i, k = 0, \ldots, m - 1. \]

To calculate the derivative, we will use the following formula for the derivative of a function of the form \( (\psi, \rho(x)) \) along the initial system (3.1) and the adjoint system (3.2):

\[ (4.21) \quad \frac{d}{dt} (\psi, \rho(x)) = -\sum_{i=0}^{m-1} u_i[r_i, \rho](x). \]
This formula implies that condition (c) is equivalent to the requirement that the matrix \((\Theta_{ik,s})_{i,k,s=0,...,m-1}\) has rank \(m - 1\), and hence that the homogeneous linear system

\[
\sum_{s=0}^{m-1} u_s \Theta_{ik,s} = 0, \quad i, k = 0, \ldots, m - 1,
\]

has a nontrivial solution, which is uniquely defined up to a constant factor. Hence at each point of \(M_0\) we can determine a tangent vector \((d\psi, dx)\) to \(M_0\) up to a constant factor. We will look for the maximal subset \(M\) of \(M_0\) with the property that \((d\psi, dx)\) is the tangent direction to \(M\) at each its point. Then through each point of \(M\) there passes a single extremal that belongs to \(M\) and hence satisfies condition (4.18). We will call such extremals the **extremals of the main stratum**.

The set \(M\) can be obtained by an iterative procedure. Let \(M_1\) consist of the points of \(M_0\) where \((d\psi, dx)\) is tangent to \(M_0\). Dealing with \(M_1\) in the same manner we obtain \(M_2\), etc. Obviously,

\[
M = \bigcap_{k=1}^{\infty} M_k.
\]

After the extremal has been found, the investigation is continued with the basis \(\hat{r}_0(x), \ldots, \hat{r}_{m-1}(x)\).

Thus for finding rigid trajectories one has to invoke not only first order conditions for extremum, but also equality type conditions which follow from quadratic necessary conditions.

Following Dmitruk, we will refer to extremals satisfying condition (4.18) as **Goh extremals**. Denote by \(G(\Psi_0)\) the set of Goh extremals in \(\Psi_0\). Since \(\Psi_0^+ \subset G(\Psi_0)\), the condition \(G(\Psi_0) \neq \emptyset\) is necessary for rigidity.

Now we state an important necessary condition in the form of a **local inequality**, i.e., an inequality that must be fulfilled at each point of the interval \([t_0, t_1]\).

Let \(\psi(\cdot) \in G(\Psi_0)\). By means of the Goh transformation one can write \(\hat{\omega}\) in the form where the control \(\bar{u}(t)\) does not appear explicitly in the integrand. Having written the form \(\hat{\omega}\) in this way by (4.14)–(4.17), collect the terms in the integrand which depend only on \(\hat{y}(t)\). Note first that since

\[
f_{ik}(t) = \psi(t)\hat{r}_{ix}(x^0(t))\hat{r}_k(x^0(t)),
\]

the function \(\hat{f}_{ik}(t)\) by the differentiation formula (4.21) can be written as

\[
\hat{f}_{ik}(t) = -\psi(t)[\hat{r}_0, \hat{r}_{ix}\hat{r}_k](x^0(t)).
\]

Using this formula and collecting similar terms, we obtain

\[
\eta(\psi(\cdot); t, \bar{y}) = \sum_{i,k=1}^{m-1} \bar{y}_i\bar{y}_k \cdot \psi(t) \cdot \left( \frac{1}{2} \hat{\rho}_{0xx}'(x^0(t)) \hat{r}_i(x^0(t))\hat{r}_k(x^0(t)) - \hat{r}_{ix}'(x^0(t)) [\hat{r}_0, \hat{r}_k](x^0(t)) + \frac{1}{2} [\hat{r}_0, \hat{r}_{ix}\hat{r}_k](x^0(t)) \right).
\]

Now we transform the expression for \(\eta(\psi(\cdot); t, \bar{y})\).

Denote by \(\rho_{ik}(t), i, k = 1, \ldots, m - 1\), the expression in parentheses in the \((i, k)\)th term. Then

\[
\eta(\psi(\cdot); t, \bar{y}) = \sum_{i,k=1}^{m-1} \bar{y}_i\bar{y}_k \psi(t)\rho_{ik}(t).
\]
Rewrite $\eta(\psi(\cdot); t, \vec{y})$ as

$$\eta(\psi(\cdot); t, \vec{y}) = \sum_{i,k=1}^{m-1} \frac{y_i y_k}{i > k} \psi(t)(\rho_{ik}(t) + \rho_{ki}(t)) + \sum_{i=1}^{m-1} y_i^2 \psi(t) \rho_{ii}(t).$$

It is not difficult to see that the following formulas hold:

$$\rho_{ik}(t) + \rho_{ki}(t) = \frac{1}{2} \left( [\hat{r}_0, \hat{r}_i], \hat{r}_k \right) (x^0(t)) + \left( [\hat{r}_0, \hat{r}_k], \hat{r}_i \right) (x^0(t)), \quad i, k = 1, \ldots, m - 1, \quad i > k;$$

$$\rho_{ii}(t) = \frac{1}{2} \left( [\hat{r}_0, \hat{r}_i], \hat{r}_i \right) (x^0(t)).$$

These formulas enable us to write $\eta(\psi(\cdot); t, \vec{y})$ in the form

$$\eta(\psi(\cdot); t, \vec{y}) = \frac{1}{2} \sum_{i,k=1}^{m-1} \frac{y_i y_k}{i > k} \psi(t) \left( [\hat{r}_0, \hat{r}_i], \hat{r}_k \right) (x^0(t)).$$

This is the form of $\eta(\psi(\cdot); t, \vec{y})$ which we aimed at.

Note that

$$\psi(t) \left( [\hat{r}_i, \hat{r}_k], \hat{r}_0 \right) (x^0(t)) = 0, \quad i, k = 0, \ldots, m - 1,$$

since

$$\psi(t) \left( [\hat{r}_i, \hat{r}_k], \hat{r}_0 \right) (x^0(t)) = \frac{d}{dt} \psi(t) \left( [\hat{r}_i, \hat{r}_k] \right) (x^0(t)),$$

and $\psi(t) \left( \hat{r}_i, \hat{r}_k \right) = 0 \mid [t_0, t_1]$ whenever $\psi(\cdot) \in G(\Psi_0)$.

It follows from results of Milyutin [3] and Dmitruk [4] that if $\psi(\cdot) \in \Psi_0^+$ then

$$\eta(\psi(\cdot); t, \vec{y}) \geq 0 \quad \forall t \in [t_0, t_1]; \quad \vec{y} \in \mathbb{R}^{m-1}.$$

Following A. V. Dmitruk, we will refer to the elements of $G(\Psi_0)$ satisfying (4.23) as Legendrian elements, and denote the set of them by $\text{Leg}(\Psi_0)$.

Thus $\Psi_0^+ \subset \text{Leg}(\Psi_0) \subset G(\Psi_0)$. Since by (4.7) the condition $\Psi_0^+ \neq \emptyset$ is necessary for rigidity, this shows that

$$\text{Leg}(\Psi_0) \neq \emptyset$$

is also a necessary condition for rigidity. This is the condition announced above.

It is easily seen from (4.22) that condition (4.25) does not depend on the choice of the basis $\hat{r}_0(x), \ldots, \hat{r}_{m-1}(x)$ and the coordinate system in $\mathbb{R}^n$.

The relation $\Psi_0^+ \subset \text{Leg}(\Psi_0)$ implies that one can replace $\Psi_0^+$ in the necessary condition (4.7) by $\text{Leg}(\Psi_0)$ to obtain again a necessary condition, though a weaker one. However, this condition is much more practical, because it is easier to verify that a function belongs to $\text{Leg}(\Psi_0)$ than to $\Psi_0^+$.

5. SUFFICIENT CONDITIONS FOR RIGIDITY

Let $x^0(t) \mid [t_0, t_1]$ be a trajectory of the differential inclusion (1.1) satisfying the condition $\text{Leg}(\Psi_0) \neq \emptyset$. To obtain sufficient conditions for rigidity we return to the problem (2.7) and the trajectory $x^0$ of the control system (2.3). Of course, first we have to choose a basis $\hat{r}_0(x), \ldots, \hat{r}_{m-1}(x)$ of $\Gamma(x)$.
According to a result by A. A. Milyutin [3] a sufficient condition for an extremum at a point $x^0$ can be stated as follows: there exists $\delta > 0$ such that

$$\sup_{\lambda \in \Lambda_+^0} \omega(\lambda; \bar{w}(\cdot)) \geq \delta \int_{t_0}^{t_1} \hat{y}^2(t) \, dt \quad \forall \bar{w}(\cdot) \in \mathcal{K}. \quad (5.1)$$

Since, according to (4.3), $\psi(\cdot) \in \Psi_0^+ \implies \lambda = (0, \psi(\cdot)) \in \Lambda_+^0$ and

$$\hat{\omega}(\psi(\cdot), \bar{w}(\cdot)) = \omega(\lambda, \bar{w}(\cdot)) \quad \forall \bar{w}(\cdot),$$

the sufficiency of condition (5.1) obviously implies the sufficiency of the following condition: for some $\delta > 0$

$$\sup_{\Psi_0^+} \hat{\omega}(\psi(\cdot), \bar{w}(\cdot)) \geq \delta \int_{t_0}^{t_1} \hat{y}^2(t) \, dt \quad \forall \bar{w}(\cdot) \in \mathcal{K}. \quad (5.2)$$

This latter condition is stated in terms of the control system (2.3) only.

However, we will use another form of sufficient condition obtained much earlier by Dmitruk [4], which can easily be carried over to our case. It is as follows: for some $\delta > 0$

$$\max_{\text{Leg}^+(\Psi_0)} \hat{\omega}(\psi(\cdot), \bar{w}(\cdot)) \geq \delta \int_{t_0}^{t_1} \hat{y}^2(t) \, dt, \quad \forall \bar{w}(\cdot) \in \mathcal{K}. \quad (5.3)$$

Since $\Psi_0^+ \subset \text{Leg}(\Psi_0)$, (5.3) is formally a weaker sufficient condition than (5.2). Actually, we will show that these two conditions are equivalent.

We will write $\psi(\cdot) \in \text{Leg}^+(\Psi_0)$ if

$$\eta(\psi(\cdot); t, \hat{y}) > 0 \quad \forall t \in [t_0, t_1] \quad \forall \hat{y} \neq 0 \quad (5.4)$$

with $\eta$ as in (4.22). Dmitruk [4] has shown that condition (5.3) implies that

$$\sup_{\text{Leg}^+(\Psi_0)} \hat{\omega}(\psi(\cdot), \bar{w}(\cdot)) = \max_{\text{Leg}^+(\Psi_0)} \hat{\omega}(\psi(\cdot), \bar{w}(\cdot)) \quad \forall \bar{w}(\cdot) \in \mathcal{K}. \quad (5.5)$$

On the other hand, $\text{Leg}^+(\Psi_0) \subset \Psi_0^+$. This implies the equivalence of conditions (5.2) and (5.3).

As shown by Dmitruk [4], the sufficient quadratic condition for an extremum can be put into a form unrelated to the variational technique. In our case this form is as follows.

Let a sequence $\{x^k\}$ of trajectories of the control system (2.3) converge to the trajectory $x^0$. This means that

$$x^k = (x^k(t), v^k(t) \mid [t_0, t_1]), \quad k = 1, 2, \ldots,$$

$$\max_{[t_0, t_1]} \left| x^k(t) - x^0(t) \right| \to 0, \quad v^k \to v^0, \quad k \to \infty, \quad (5.5)$$

$$\operatorname{esssup}_{[t_0, t_1]} \left| u^k(t) - u^0(t) \right| \to 0, \quad k \to \infty. \quad (5.6)$$

Let, furthermore,

$$x^k(t_0) = x^0(t_0). \quad (5.6)$$

Put $\delta y^k(t) = \int_{t_0}^{t} (u^k(\tau) - u^0(\tau)) \, d\tau$. Assume that

$$\delta y^k(t) \neq 0 \mid [t_0, t_1] \quad (5.7)$$

for sufficiently large $k$. Then (5.3) is equivalent to the condition

$$\liminf_{k \to \infty} \frac{\left| x^k(t_1) - x^0(t_1) \right|}{\int_{t_0}^{t_1} (\delta y^k(t))^2 \, dt} > 0 \quad (5.8)$$
for any sequence \( x \) satisfying conditions (5.5), (5.6), and (5.7).

This result, which is called a deciphering in the theory of higher order conditions, immediately implies that condition (5.3) does not depend on the choice of the basis \( \hat{r}_0(x), \ldots, \hat{r}_{m-1}(x) \), nor on the choice of coordinates in \( \mathbb{R}^n \). Thus condition (5.3) is an invariant sufficient condition for rigidity.

The condition \( \text{Leg}^+(\Psi_0) \neq \emptyset \), concomitant with the sufficient condition (5.3), is in itself a sufficient condition for local rigidity.

Let us prove this assertion. Let \( \psi_*(\cdot) \in \text{Leg}^+(\Psi_0) \). Then \( \eta(\psi_*(\cdot); t, \bar{y}) \geq \delta(t)\bar{y}^2, \delta(t) > 0 \mid [t_0, t_1] \). By continuity we obtain

\[
(5.9) \quad \eta(\psi_*(\cdot); t, \bar{y}) > \delta_0 \bar{y}^2, \quad \delta_0 > 0 \mid [t_0, t_1].
\]

Let \( \Delta \) be a subinterval of \([t_0, t_1]\). We will denote by \(|\Delta|\) the length of \( \Delta \) and by \( t_1(\Delta), t_r(\Delta) \) the endpoints of \( \Delta \).

We will show that condition (5.3) is fulfilled for all \( \Delta \) with sufficiently small \(|\Delta|\). According to (3.14) and (3.15), for given \( \Delta \) the cone of critical variations \( K_\Delta \) is defined by the conditions

\[
\ddot{x} = \hat{r}'_{0x}(x^0(t))\dot{x} + \bar{v}\hat{r}_0(x^0(t)) + \sum_{i=1}^{m-1} \bar{u}_i \hat{r}_i(x^0(t)),
\]

\[
\dot{v} = 0, \quad \bar{x}(t_1(\Delta)) = \bar{x}(t_r(\Delta)) = 0.
\]

Putting \( \bar{y} = \int_{t_1(\Delta)}^{t_r(\Delta)} \bar{u}(t) \, dt \) and applying the Goh transformation, we conclude that \((\xi(t), \bar{v}, \bar{y}(t)) \in K_\Delta \) if and only if

\[
(5.10) \quad \dot{\xi} = \hat{r}'_{0x}(x^0(t))\xi + \bar{v}\hat{r}_0(x^0(t)) + \sum_{i=1}^{m-1} \bar{y}_i(t)[\hat{r}_0, \hat{r}_i](x^0(t)), \quad \dot{\bar{v}} = 0,
\]

\[
\xi(t_1(\Delta)) = 0, \quad \xi(t_r(\Delta)) + \sum_{i=1}^{m-1} \bar{y}_i(t_r(\Delta))\hat{r}_i(x^0(t_r(\Delta))) = 0.
\]

This implies that

\[
\xi(t_r(\Delta)) = \bar{v} \int_{\Delta} \hat{r}_0(x^0(t)) \, dt + o(|\bar{v}| \cdot |\Delta|) + o\left(\sqrt{\int_{\Delta} \bar{y}^2 \, dt}\right)
\]

as \(|\Delta| \to 0\). Since \( \hat{r}_0(x), \ldots, \hat{r}_{m-1}(x) \) are linearly independent, the boundary condition at the point \( t_r(\Delta) \) implies

\[
(5.11) \quad |\bar{v}| \cdot |\Delta| = o\left(\sqrt{\int_{\Delta} \bar{y}^2 \, dt}\right) \quad \text{as} \quad |\Delta| \to 0.
\]

Hence

\[
(5.12) \quad \max_{\Delta} |\bar{\xi}(t)| = o\left(\sqrt{\int_{\Delta} \bar{y}^2(t) \, dt}\right) \quad \text{as} \quad |\Delta| \to 0.
\]

Finally, from (5.12) we obtain

\[
(5.13) \quad |\bar{y}(t_r(\Delta))| = o\left(\sqrt{\int_{\Delta} \bar{y}^2(t) \, dt}\right) \quad \text{as} \quad |\Delta| \to 0.
\]
Consider the form \( \tilde{\omega}(\psi(\cdot), \bar{w}(\cdot)) \) on \( \mathcal{K}_\Delta \). Since \( \psi(\cdot) \in G(\Psi_0) \), the form \( \tilde{\omega} \), according to (4.14)–(4.17), can be rewritten in the variables \( \xi(t), \bar{y}(t) \). Taking (5.11), (5.12), and (5.13) into account, we obtain that for any \( \bar{w}(\cdot) \in \mathcal{K}_\Delta \)

\[
\tilde{\omega}(\psi(\cdot), \bar{w}(\cdot)) = o \left( \int_{\Delta} \bar{y}^2(t) \, dt \right) + \int_{\Delta} \eta(\psi(\cdot); t, \bar{y}(t)) \, dt \quad \text{as} \quad |\Delta| \to 0.
\]

Taking (5.9) into account, we obtain that for all sufficiently small \( \Delta \)

\[
(5.14) \quad \tilde{\omega}(\psi(\cdot), \bar{w}(\cdot)) \geq \int_{\Delta} \delta_\epsilon \bar{y}^2(t) \, dt \quad \forall \bar{w}(\cdot) \in \mathcal{K}.
\]

Thus we have established that condition (5.3) is fulfilled for all sufficiently small \( \Delta \). Therefore the trajectory \( x^0(t) \) is rigid for all sufficiently small \( \Delta \), i.e., locally rigid. Thus our assertion is proved.

The condition \( \text{Leg}^+(\Psi_0) \neq \emptyset \) contains some additional information. Namely, it turns out that any extremal \( \psi(t), x^0(t), u^0_0(t), u^0(t) \mid [t_0, t_1] \) such that \( \psi(\cdot) \in \text{Leg}^+(\Psi_0) \) belongs to the main stratum.

Indeed, let \( \psi(\cdot) \in \text{Leg}^+(\Psi_0) \). Then (4.22) and (5.9) imply that the form

\[
\sum_{i,k=1}^{m-1} \bar{y}_i \bar{y}_k \psi(t) [\bar{r}_0, \bar{r}_i, \bar{r}_k] (x^0(t))
\]

is positive definite for any \( t \in [t_0, t_1] \). Then \( \|a_{ik}(t)\| \neq 0 \) for any \( t \in [t_0, t_1] \), where

\[
a_{ik}(t) = \psi(t) [\bar{r}_0, \bar{r}_i, \bar{r}_k] (x^0(t)).
\]

Since by (4.23)

\[
\psi(t) [\bar{r}_i, \bar{r}_k, \bar{r}_0] (x^0(t)) = 0, \quad i, k = 0, \ldots, m - 1,
\]

this implies that

\[
\dim \text{Lin} \left( \{\Theta_{ik}(t)\}_{i,k=0, \ldots, m-1} \right) = m - 1,
\]

where the vector \( \Theta_{ik}(t) \) is defined by (4.19). (We write \( \dim \) for the dimension of a linear subspace and \( \text{Lin} \) for the linear span of a collection of vectors.)

This shows that the extremal

\[
\psi(t), x^0(t), u^0_0(t), u^0(t) \mid [t_0, t_1]
\]

of the control system (2.2) is an extremal of the main stratum.

As we pointed out above, the condition \( \text{Leg}^+(\Psi_0) \neq \emptyset \) is a consequence of (5.3). Therefore the assertion just obtained implies that the sufficient condition (5.3) can be realized only on the trajectories which are state components of extremals of the main stratum.

Like (5.3), the condition \( \text{Leg}^+(\Psi_0) \neq \emptyset \) has a meaning independent of variational technique. The results by A. V. Dmitruk [7], as applied to our case, imply that the condition \( \text{Leg}^+(\Psi_0) \neq \emptyset \) is equivalent to the following one.

Let \( \{x^k\}_{k=1,2} \) be a sequence of trajectories of the control system (2.3) converging to a trajectory \( x^0 \) in the sense of (5.5). Suppose the following additional requirement is fulfilled:

\[
\int_{t_0}^{t_1} |\delta y^k(t)| \, dt = o \left( \sqrt{\int_{t_0}^{t_1} (\delta y^k(t))^2 \, dt} \right).
\]
(Dmitruk calls such sequences *Legendrian*.) Then for any Legendrian sequence the following inequality must hold:

\[
\liminf_{k \to \infty} \frac{|x^k(t_1) - x^0(t_1)|}{\int_{t_0}^{t_1} (\delta y^k(t))^2 \, d} > 0.
\]

This shows that the property \( \text{Leg}^+ (\Psi_0) \neq \emptyset \) does not depend on the choice of the basis \( \vec{r}_0(x), \ldots, \vec{r}_{m-1}(x) \) and the coordinate system in \( \mathbb{R}^n \). For this reason, when dealing with this and other invariant properties, we will speak of extremals of the differential inclusion \( (1.1) \), that is, of pairs \( \psi(t), x(t) \), without specifying in which control system, \( (2.2) \) or \( (3.2) \), they are considered.

The condition \( \text{Leg}^+ (\Psi_0) \neq \emptyset \) as a sufficient condition for rigidity can be weakened without losing this feature. Namely, we can require \( \text{Leg}^+ (\Psi_0(\Delta)) \neq \emptyset \) to be fulfilled for any sufficiently small interval \( \Delta \), where \( \Psi_0(\Delta) \) are normalized adjoint components of extremals defined on \( \Delta \). The proof does not need any changes.

Accordingly, a necessary condition for local rigidity is that

\[
\text{Leg} (\Psi_0(\Delta)) \neq \emptyset
\]

for any sufficiently small interval \( \Delta \).

However in this latter condition the intervals \( \Delta \) cannot be replaced by the entire interval \([t_0, t_1]\), as might be expected. Contrary to Theorem 2 in [9, p. 1475], the condition \( \text{Leg}^+ (\Psi_0) \neq \emptyset \) is not necessary for local rigidity. We will prove this by an example in the next section.

### 6. The Case of Two-Dimensional \( \Gamma(x) \)

In the previous section we have shown that any extremal \( \psi(t), x^0(t) \) with \( \psi(t) \in \text{Leg}^+ (\Psi_0) \) is an extremal of the main stratum. If \( \Gamma(x) \) is two-dimensional, a converse assertion is true in a certain sense.

**Theorem 6.1.** Let a trajectory \( x^0(t) | [t_0, t_1] \) be the state component of an extremal of the main stratum. If \( \Gamma(x) \) is two-dimensional, a converse assertion is true in a certain sense.

**Proof.** Let \( \psi(t), x^0(t) | [t_0, t_1] \) be an extremal of the main stratum. Introduce a basis \( \vec{r}_0(x), \vec{r}_1(x) \) in \( \Gamma(x) \) and consider the extremal \( \psi(t), x^0 \) of the control system \( (2.3) \). By the definition \( (4.19) \), the extremal belongs to the main stratum in \( (2.3) \) if the expressions

\[
\psi(t) \left[ [\vec{r}_0, \vec{r}_1], \vec{r}_0 \right] (x^0(t)), \quad \psi(t) \left[ [\vec{r}_0, \vec{r}_1], \vec{r}_1 \right] (x^0(t))
\]

do not vanish on \([t_0, t_1]\) simultaneously. But by \( (4.23) \) the former equals zero at any point of \([t_0, t_1]\). Hence the latter is everywhere different from zero and by continuity retains its sign on \([t_0, t_1]\).

Since \( \Psi_0 = -\Psi_0 \), changing the sign of \( \psi(\cdot) \) if necessary, we can assume that

\[
\psi(t) \left[ [\vec{r}_0, \vec{r}_1], \vec{r}_1 \right] (x^0(t)) > 0 \quad \forall \, t \in [t_0, t_1].
\]

By \( (4.22) \),

\[
\eta(\psi(\cdot); t, \vec{y}) = \frac{1}{2} \vec{y}^T \psi(t) \left[ [\vec{r}_0, \vec{r}_1], \vec{r}_1 \right] (x^0(t)).
\]

Therefore

\[
\eta(\psi(\cdot); t, \vec{y}) > 0 \quad \forall \, t \in [t_0, t_1], \quad \vec{y} \neq 0.
\]

By definition, this means that \( \psi(\cdot) \in \text{Leg}^+ (\Psi_0) \). The proof is completed.

This theorem and the sufficient condition for local rigidity imply
Theorem 6.2. Any trajectory $x^0(t)$ of the differential inclusion (1.1) which is the state component of an extremal of the main stratum is locally rigid.

The hypothesis of Theorem 6.2 can be easily reduced in a natural way. It is sufficient to require that for any point $t \in [t_0, t_1]$ there is a neighborhood in which $x^0(t)$ is the state component of an extremal of the main stratum. Then the conclusion of the theorem remains unchanged. In [8] a sufficient condition for local rigidity is given which is considerably stronger than that of Theorem 6.2. It is required in [8] that the distribution determines an Engel structure. We do not impose this condition. At the end of the paper we will give the corresponding example.

Next we consider four examples.

Example 1. Consider the control system

\begin{equation}
\dot{z} = u_0 \cdot \frac{1}{2} (y^2 - x^2), \quad \dot{x} = u_0y, \quad \dot{\tau} = u_0, \quad \dot{y} = u_1.
\end{equation}

Here $z$, $x$, $\tau$, $y$ are scalar variables. Setting $w = (z, x, \tau, y)$ and

\begin{align*}
r_0(w) &= \left( \frac{1}{2} (y^2 - x^2), y, 1, 0 \right), \quad r_1(w) = (0, 0, 0, 1),
\end{align*}

we can rewrite (6.1) as

\begin{equation}
\dot{w} = u_0 r_0(w) + u_1(w) r_1(w).
\end{equation}

Let $\Gamma(w) = \text{Lin} (r_0(w), r_1(w))$. It is easily seen that $\dim \Gamma(w) = 2$ for any $w$.

Thus the system (6.1) is equivalent to the following two-dimensional differential inclusion in four-dimensional space:

\begin{equation}
\dot{w} \in \Gamma(w).
\end{equation}

The control system (6.1) is closely related to the well-known problem of calculus of variations

\begin{equation}
\frac{1}{2} \int (\dot{x}^2 - \dot{\tau}^2) \, dt \to \min.
\end{equation}

In general, variational problems convex with respect to $\dot{x}$ may give rise to differential inclusions of type (1.1).

Let

\begin{align*}
c &= [r_0, r_1](w), \quad p = [c, r_0](w), \quad q = [c, r_1](w).
\end{align*}

Then

\begin{equation}
c = (y, 1, 0, 0), \quad p = (x, 0, 0, 0), \quad q = (1, 0, 0, 0).
\end{equation}

This shows that

\[ \dim (\text{Lin}(r_0(w), r_1(w), c(w), q(w))) = 4 \quad \forall \, w. \]

Therefore the control system (6.1) has no first integrals, since the vectors $r_0(w)$, $r_1(w)$, $c(w)$, $q(w)$ for any $w$ form a basis in $\mathbb{R}^4$.

Let us find the Goh extremals of the system. The adjoint equation has the form

\[ \dot{\psi} = -u_0 r_0'(w) - u_1 r_1'(w). \]

Here $\psi = (\psi_z, \psi_x, \psi_\tau, \psi_y) \in \mathbb{R}^4$.

Putting the expressions for $r_0(w)$, $r_1(w)$ into the adjoint equation, we obtain

\begin{equation}
\dot{\psi}_z = 0, \quad \dot{\psi}_x = u_0 \psi_z x, \quad \dot{\psi}_\tau = 0, \quad \dot{\psi}_y = -u_0 (\psi_z y + \psi_x).
\end{equation}
It is clear from these equations that $\psi_x = \text{const}$, $\psi_r = \text{const}$ on an extremal. A Goh extremal must satisfy the conditions

$$\psi r_0(w) = 0, \quad \psi r_1 = 0, \quad \psi c(w) = 0,$$

as well as the nontriviality condition $\psi \not= 0$.

Since $r_0(w), r_1, c(w), q(w)$ is a basis of $\mathbb{R}^4$, these conditions imply that $\psi q(w) \not= 0$ on the Goh extremal. Hence the Goh extremal belongs to the main stratum. By Theorem 6.2 this implies that the state component of the Goh extremal is locally rigid.

By (6.4) the condition $\psi q \not= 0$ means that $\psi_x \not= 0$. We continue our investigation, putting $\psi_x = -1$. Then we can easily find the extremals of the main stratum. We have

$$u_0 \psi p(w) + u_1 \psi q(w) = 0.$$

Taking (6.4) into account, this equality becomes

(6.6) $$u_0 x + u_1 = 0.$$ Using this equality, we can express $u_0$ and $u_1$ through $w$.

For example, we can set

(6.7) $$u_0 = 1, \quad u_1 = -x.$$ From these equalities and the condition $\psi c = 0$ we finally obtain $\psi_x = -1$, $\psi_x = y$, $\dot{x} = y$, $\dot{y} = -x$. Moreover, from the conditions $\psi r_0 = 0$, $\psi r_1 = 0$ we see that

(6.8) $$\frac{1}{2}(y^2 + x^2) + \psi_x = 0, \quad \psi_y = 0.$$ Thus we have obtained the complete description of the extremals of the main stratum.

Take the basis $r_0(w), r_1(w)$ by putting

(6.9) $$r_0 = \left(\frac{1}{2}(y^2 - x^2), y, 1, -x\right), \quad r_1 = (0, 0, 0, 1).$$ Then the control system (3.12) becomes

(6.10) $$\dot{w} = v \tilde{r}_0(w) + u_1 \tilde{r}_1(w), \quad \dot{v} = 0.$$ The trajectory $\mathcal{X}^0$ is determined by the equalities

(6.11) $$v^0 = 1, \quad u_1^0(t) = 0 \quad \forall t.$$ Putting the expressions (6.9) for $\tilde{r}_0, \tilde{r}_1$ into (6.10) yields

(6.12) $$\dot{z} = v \frac{1}{2}(y^2 - x^2), \quad \dot{x} = vy, \quad \dot{r} = v, \quad \dot{y} = -vx + u_1, \quad \dot{v} = 0.$$ Then the linearized system becomes

(6.13) $$\dot{z} = y\ddot{y} - x\ddot{x} + \ddot{v} \frac{1}{2}(y^2 - x^2), \quad \dot{x} = \ddot{y} + \ddot{v} y, \quad \ddot{r} = \ddot{v}, \quad \dot{y} = -\ddot{x} - \ddot{v} x + \ddot{u}_1, \quad \dot{v} = 0.$$ For the determination of the critical cone on the interval $[0, T]$ the equations (6.13) should be supplemented by the condition $\ddot{w}(0) = \ddot{w}(T) = 0$.

From the third and fifth equations of the linearized system we derive

$$\ddot{w}(\cdot) \in \mathcal{K} \implies \ddot{v} = 0, \quad \ddot{r} = 0.$$
Thus there remain three equations:

\[
\begin{align*}
\dot{z} & = y\dot{y} - x\dot{x}, \\
\dot{x} & = \dot{y}, \\
\dot{y} & = -\dot{x} + \ddot{u}_1.
\end{align*}
\]

However we need not consider the first equation since its right-hand side is the derivative of \(y\dot{x}\), so that the boundary conditions on \(z\) are fulfilled whenever the boundary conditions on \(\dot{z}\) are fulfilled.

According to (6.8), the set \(G(\Psi_0)\) contains a single function, up to its sign.

The form \(\varpi\) is given by

\[
\varpi = \int_0^T \frac{1}{2}(\dot{y}^2 - \dot{x}^2) \, dt.
\]

This expression does not involve \(\ddot{u}_1\).

According to condition (5.3) the form \(\varpi\) should be compared with \(\int_0^T \dot{h}^2(t) \, dt\), where \(\dot{h} = \ddot{u}_1(t), \dot{h}(0) = 0\). However it is easily seen that the functionals \(\int_0^T \dot{y}^2(t) \, dt\) and \(\int_0^T \dot{h}^2(t) \, dt\) are equivalent.

Therefore we can replace \(h\) by \(\dot{y}\) and compare the form \(\varpi\) with \(\int_0^T \dot{y}^2 \, dt\).

It is easily seen that the conditions \(\dot{y}(0) = \dot{y}(1) = 0\) can be dropped without affecting the derivations, thanks to condition (5.3). Thus we arrive at the investigation of the form \(\varpi\) on the solutions of the equation \(\dot{x} = \dot{y}\) satisfying the boundary conditions \(x(0) = x(T) = 0\).

This form has been thoroughly studied. The sufficient condition (5.3) is fulfilled if and only if \(T < \pi\). Thus rigid trajectories are state components of extremals of the main stratum if they are considered on a time interval smaller than \(\pi\). If the time interval is greater than \(\pi\), then the necessary condition (4.7) fails because the form \(\varpi\) changes its sign on \(K\). When the time interval equals \(\pi\) the necessary condition (4.7) is satisfied, but the sufficient condition (5.3) fails.

**Example 2.** In this example we consider a single trajectory and investigate it for local rigidity.

Let \(x = (x_0, x_1, x_2)\) and let \(e_0, e_1, e_2\) be the basis in \(\mathbb{R}^3\) such that \(e_0 = (1, 0, 0), e_1 = (0, 1, 0), e_2 = (0, 0, 1)\). We specify the distribution \(\Gamma(x)\) by

\[
\begin{align*}
\gamma_0(x) &= e_0, \\
\gamma_1(x) &= \left(1 + \frac{x_0^2}{2}\right)e_1 + Ae,
\end{align*}
\]

where \(A\) is the linear transformation specified by the conditions \(Ae_0 = 0, Ae_1 = e_2, Ae_2 = 0\). We consider the trajectory \(u_0 = 1, u_1 = 0\), which passes through the point \(x = 0\). Then the trajectory has the form

\[
x_0 = t, \quad x_1 = x_2 = 0.
\]

Denote it by \(\mathcal{R}^0\). Let \(c = [\gamma_0, \gamma_1], q = [c, \gamma_1]\). Then

\[
\begin{align*}
c &= x_0e_1, \\
q &= x_0 Ae_1 = x_0e_2.
\end{align*}
\]

We will show that \(\mathcal{R}^0\) on each interval \([t_0, t_1]\) is the trajectory component of an extremal whose adjoint component is uniquely defined up to a constant factor.

Indeed, the extremality conditions yield \(\psi(t) = \text{const}\) (the adjoint equation). The conditions \(\psi_0 = 0, \psi_1 = 0 \mid \mathcal{R}^0\) imply \(\psi = \Theta e_2\), where \(\Theta = \text{const}\). Hence \(\psi_0\) consists of two elements

\[
\psi = \pm e_2.
\]

It is clear that \(\psi c = 0\), which implies that the extremal is a Goh extremal.
It follows from (6.17) that in a neighborhood of each point \( t \neq 0 \) the extremal belongs to the main stratum. Thus the trajectory \( x^0 \) is locally rigid in a neighborhood of each point \( t \neq 0 \).

However it is not rigid in a neighborhood of \( t = 0 \), because in a neighborhood of zero \( \psi q = \pm t \); therefore on each interval containing zero as an interior point we have \( \text{Leg}^+(\Psi_0) \neq \emptyset \). This means that the necessary condition for rigidity is not fulfilled.

This example shows that even a minor (at a single point) violation of the requirement that the extremal belongs to the main stratum can cause the loss of local rigidity.

Next we consider an example related to local rigidity. Recall the conditions of local rigidity stated in the end of Section 5.

Let \( x^0(t) \) be a locally rigid trajectory of the differential inclusion (1.1). Then for any sufficiently small interval \( \Delta \) the following condition must be fulfilled:

\[
\text{Leg}(\Psi_0(\Delta)) \neq \emptyset.
\]

Further, let \( x^0(t) \) be a trajectory. Then the condition

\[
\text{Leg}^+(\Psi_0(\Delta)) \neq \emptyset
\]

for any sufficiently small interval \( \Delta \) ensures that the trajectory \( x^0(t) \) is locally rigid.

It is natural to ask whether the necessary condition (6.18) can be strengthened by substituting \( \Psi_0 \) for \( \Psi_0(\Delta) \) in it. A. Agrachev and A. Sarychev [9], Theorem 2(a), p. 1472, answered this question in the affirmative, if we restate their conclusion in our terms. However the following example shows that this is not the case.

**Example 3.** Consider \( \mathbb{R}^5 \) with elements \( x = (x_0, \ldots, x_4) \). Let \( e_0, \ldots, e_4 \) be the standard basis in \( \mathbb{R}^5 \), i.e., \( e_i = (\delta_{i0}, \ldots, \delta_{i4}) \), \( i = 0, 1, \ldots, 4 \). The distribution \( \Gamma(x) \) will be specified by the basis

\[
r_0(x) = e_0, \quad r_1(x) = \sin x_0 e_1 + \cos x_0 e_2 + x_1 e_3 + x_2 e_4.
\]

It is easily seen that \( \dim \Gamma(x) = 2 \) for any \( x \). We will show that \( \Gamma(x) \) at any point \( x \) is bracket generating the whole space \( \mathbb{R}^5 \).

Let \( c = [r_0, r_1] \), \( q = [c, r_1] \), \( s = [r_0, q] \). Then

\[
\begin{align*}
c &= - r_0'_{x_0} = - \cos x_0 e_1 + \sin x_0 e_2, \\
q &= - r_1'_{x_0} \cdot c = + \cos x_0 e_3 - \sin x_0 e_4, \\
s &= - q'_{x_0} = - \sin x_0 e_3 - \cos x_0 e_4.
\end{align*}
\]

It is easily seen from (6.20) that

\[
\text{Lin}(r_0, r_1, c, q, s) = \mathbb{R}^5 \quad \forall x,
\]

which proves our assertion.

Consider the control system

\[
\dot{x} = u_0 r_0 + u_1 r_1(x)
\]

and take the trajectory \( x^0 \) specified by the conditions

\[
u_0 = 1, \quad u_1 = 0, \quad x(0) = 0.
\]

The basis \( r_0, r_1 \) with respect to this trajectory has the properties of the basis \( \tilde{r}_0, \tilde{r}_1 \). We will show that on each interval \( \Delta \) this trajectory can be completed to an extremal.

Indeed, by the adjoint equation (3.2), \( \psi = \text{const} \). The conditions \( \psi \tau_0 = \psi \tau_1(x(t)) = 0 \) yield \( \psi_0 = 0 \) and \( \sin t \psi_1 + \cos t \psi_2 = 0 \), whence \( \psi_0 = \psi_1 = \psi_2 = 0 \). Therefore

\[
\psi \in \Psi_0(\Delta) \iff \psi_0 = \psi_1 = \psi_2 = 0, \quad \psi_3^2 + \psi_4^2 = 1.
\]
It is seen from (6.22) that $\Psi_0(\Delta)$ does not depend on $\Delta$. Hence we will write simply $\Psi_0$. It is easily seen that $\psi c = 0$ for any $\psi \in \Psi_0$. Therefore $\Psi_0 = G(\Psi_0)$.

Next, we write down $\eta(\psi; \bar{y}, t), \psi \in \Psi_0$. According to Section 4 we obtain

\[(6.23) \quad \eta = \frac{1}{2} \psi q \bar{y}_1^2 = \frac{1}{2} (\cos t \cdot \psi_3 - \sin t \cdot \psi_4) \bar{y}_1^2.\]

It is clear from this and (6.22) that for $|\Delta| > \pi$ one always has $\text{Leg} (\Psi_0(\Delta)) = \emptyset$, while for $|\Delta| < \pi$ one always has $\text{Leg}^+ (\Psi_0(\Delta)) \neq \emptyset$, where $|\Delta|$ is the length of $\Delta$.

The latter assertion implies that the trajectory $\gamma^0$ is locally rigid on each interval $\Delta$. The former assertion implies that one cannot dispense with locality in condition (6.18).

One can ask whether each locally rigid trajectory is the state component of an extremal defined on the same interval as the trajectory itself. Unfortunately, this is also not the case. We can see this by an example, somewhat more complicated than the previous one.

**Example 4.** Take the following basis of $\Gamma(x)$:

\[r_0 = e_0,\]
\[r_1(x) = \sin x_0 \cdot e_1 + \cos x_0 \cdot e_2 + f_1(x_0) e_3 + f_2(x_0) e_4 + x_1 e_3 + x_2 e_4.\]

Here
\[f_2(x_0) = \begin{cases} x_0^4, & x_0 \leq 0, \\ 0, & x_0 \geq 0, \end{cases} \quad f_1(x_0) = f_2 \left(-\left(x_0 - \frac{3}{2} \pi\right)\right).\]

Let $\gamma^0$ be a trajectory of the control system on which $u_0 = 1, u_1 = 0, x(0) = 0$. We consider it on the interval $\Delta^0 = [-\frac{\pi}{4}, \frac{7}{4} \pi]$. We will show that $\Psi_0(\Delta^0) = \emptyset$. Indeed, it follows from the adjoint equation that $\bar{\psi} = \text{const}$. Then it is easy to infer from the conditions $\psi r_0 = 0$ and $\psi r_1(x(t)) = 0 \mid \Delta^0$ that $\bar{\psi} = 0$. Thus $\Psi_0(\Delta^0) = \emptyset$. In other words, the trajectory $\gamma^0$ on the interval $\Delta^0$ is not the state component of an extremal.

At the same time, we will show that the trajectory $\gamma^0$ on $\Delta^0$ is rigid. We have
\[c = -\cos x_0 \cdot e_1 + \sin x_0 \cdot e_2 - f'_1(x_0) e_3 - f'_2(x_0) e_4,\]
\[q = \cos x_0 \cdot e_3 - \sin x_0 \cdot e_4.\]

Note that the basis $r_0, r_1(x)$ with respect to the trajectory $\gamma^0$ has the properties of the basis $\hat{r}_0, \hat{r}_1$.

Let $\Delta_1 = [-\frac{\pi}{4}, \frac{\pi}{4}]$. It is easily seen that
\[\Psi_0(\Delta_1) = G(\Psi_0(\Delta_1)) = \{\bar{\psi} \mid \bar{\psi} = \pm e_3\}.\]

Then taking $\psi = e_3$ we obtain $\eta = \frac{1}{2} \cos t \cdot \bar{y}_1^2$, whence $\eta > 0 \mid \Delta_1$. Therefore $\text{Leg}^+ (\Psi_0(\Delta_1)) \neq \emptyset$, which implies that the trajectory $\gamma^0$ is locally rigid on $\Delta_1$.

Let $\Delta_2 = [0, \frac{3}{2} \pi]$. On $\Delta_2$ our system coincides with the system in the previous example; hence the trajectory is locally rigid on $\Delta_2$.

Finally, let $\Delta_3 = [\frac{5}{4} \pi, \frac{7}{4} \pi]$. We have
\[\Psi_0(\Delta_3) = G(\Psi_0(\Delta_3)) = \{\bar{\psi} \mid \bar{\psi} = \pm e_4\}.\]

Taking $\psi = e_4$, we obtain
\[\eta = -\frac{1}{2} \sin t \cdot \bar{y}_1^2.\]

Then $\eta > 0 \mid \Delta_3$; hence $\text{Leg}^+ (\Psi_0(\Delta_3)) \neq \emptyset$. Therefore the trajectory $\gamma^0$ is locally rigid on $\Delta_3$ as well. The intervals $\Delta_1, \Delta_2$, and $\Delta_3$ overlap and cover the interval $\Delta^0$. Thus for each point of $\Delta^0$ one can find a neighborhood in which $\gamma^0$ is a rigid trajectory. Therefore $\gamma^0$ is locally rigid on $\Delta^0$. 

\[\text{RIGIDITY AND OPTIMAL CONTROL} \quad 109\]
It is seen from Examples 3 and 4 that conditions for local rigidity must be of local nature themselves.

7. Example of a Distribution of an Arbitrary Dimension

Let the basis of $\Gamma(x)$ be of the form:

$$r_0 = \text{const} = r, \quad r_i(x) = A_i x, \quad i = 1, \ldots, m - 1.$$  

Here $A_i : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. We assume that

$$[A_i, A_k]r = 0, \quad i, k = 1, \ldots, m - 1,$$

where $[A, B] = AB - BA$.

Write the differential inclusion $\dot{x} \in \Gamma(x)$ as the control system

$$\dot{x} = u_0r_0 + \sum_{i=1}^{m-1} u_i A_i x.$$  

Let us find the extremals of the main stratum for the control system (7.3). The local extremality conditions, i.e., the maximality conditions, have the form

$$\psi r = 0, \quad \psi A_i x = 0, \quad i = 1, \ldots, m - 1.$$  

The Goh conditions are

$$\psi [r_i, r_k](x) = 0, \quad i, k = 0, \ldots, m - 1.$$  

By (7.1) we have

$$[r_0, r_i](x) = -A_i r, \quad i = 1, \ldots, m - 1,$$

$$[r_i, r_k](x) = [A_i, A_k] x, \quad i, k = 1, \ldots, m - 1.$$  

Hence the Goh conditions become

$$\psi A_i r = 0, \quad i = 1, \ldots, m - 1,$$

$$\psi [A_i, A_k] x = 0, \quad i, k = 1, \ldots, m - 1.$$  

Differentiating the equalities (7.5) with respect to time, by (4.20) we obtain the system of equations for $u_0, u_1, \ldots, u_{m-1}$:

$$\sum_{k=0}^{m-1} u_k \psi [r_0, r_i, r_k](x) = 0, \quad i, k = 0, \ldots, m - 1,$$

$$\sum_{k=0}^{m-1} u_k \psi [r_j, r_s, r_k](x) = 0, \quad j, s = 1, \ldots, m - 1.$$  

Taking into account (7.1) and assumption (7.2), we obtain the following system:

$$\sum_{k=0}^{m-1} u_k \psi [A_k, A_i] r = 0, \quad i = 1, \ldots, m - 1;$$

$$\sum_{k=1}^{m-1} u_k \psi [A_j, A_k] x = 0, \quad j, s = 1, \ldots, m - 1.$$  

Since $u_0$ does not enter into the system, for an extremal passing through the point $(\psi, x)$ to belong to the main stratum it is necessary that the system has a unique solution, i.e., that $u_i = 0, i = 1, \ldots, m - 1$, is its only solution.
Normalizing \( u_0 \) by the condition \( u_0 = 1 \), we obtain that the control on an extremal of the main stratum is

\[
(7.8) \quad u_0 = 1, \quad u_i = 0, \quad i = 1, \ldots, m - 1.
\]

Thus the trajectories which correspond to extremals of the main stratum are the solutions of the system

\[
(7.9) \quad \dot{x} = r.
\]

On the other hand, the adjoint components of an extremal of the main stratum satisfy the equation

\[
(7.10) \quad \dot{\psi} = 0.
\]

Now we can give the extremality and the Goh conditions a more concrete form.

It follows from (7.4) and (7.5) that the extremality conditions are equivalent to the conditions

\[
(7.11) \quad \psi r = 0, \quad \psi A_i r = 0, \quad \psi A_i x = 0, \quad i = 1, \ldots, m - 1.
\]

The last condition is fulfilled whenever it is fulfilled at some time point. Now the Goh conditions become

\[
(7.12) \quad \psi[A_j, A_s] x = 0, \quad j, s = 1, \ldots, m - 1.
\]

As follows from (7.2), it suffices to verify this for some time point as well. We normalize \( \psi \) by the condition \( |\psi| = 1 \). For a given \( x \) let \( \Psi_0(x) \) be the set of normalized \( \psi \) satisfying (7.11). Denote by \( G(x) \) the subset of \( \Psi_0(x) \) which consists of \( \psi \) satisfying (7.12).

Using the possibility of time translation, we will consider all trajectories of the family satisfying the equation (7.9) on a time interval with the left end-point at zero.

Let \( x_0 = x(0), \ x_1 = x(T) \), where \([0, T]\) is the interval on which we consider the trajectory. Obviously, \( \Psi_0(x(t)) \) and \( G(x(t)) \) do not change on a trajectory of this family. If a trajectory joins the points \( x_0 \) and \( x_1 \), this uniquely determines \( T \). For this reason we will denote the trajectory by \( \{x_0; x_1\} \).

Let us investigate the trajectories of the family for rigidity. For that we will use the conditions obtained in Sections 4 and 5. First of all, according to Section 2, we have to pass to the basis \( \tilde{r}_0(x), \ldots, \tilde{r}_{m-1}(x) \), which is related to the trajectory by requirements (2.1). But this is not needed for our family, because the initial basis already has the required properties for any trajectory of the family. The next step is to pass to the control system (2.3). It has the form

\[
(7.13) \quad \dot{x} = v r + \sum_{1}^{m-1} u_i A_i x, \quad \dot{v} = 0.
\]

Each trajectory of the family is a solution of the system (7.13) with \( v = 1, \ u(t) = 0 \).

Let us find the cone of critical variations \( \mathcal{K}(x_0, x_1) \) for a trajectory \( \{x_0, x_1\} \). By definition, \( \bar{w}(t) = (\bar{x}(t), \bar{v}, \bar{u}(t)) \in \mathcal{K}(x_0, x_1) \) if and only if

\[
(7.14) \quad \dot{x} = \bar{v} r + \sum_{1}^{m-1} \bar{u}_i A_i x, \quad \dot{v} = 0,
\]

\[
(7.15) \quad \bar{x}(0) = 0, \quad \bar{x}(T) = 0.
\]
Rewrite $\mathcal{K}(x_0, x_1)$ by means of the Goh transformation, which in our case is

\begin{equation}
\dot{y} = u, \quad \bar{y}(0) = 0, \quad \bar{x} = \bar{\xi} + \sum_{i=1}^{m-1} \bar{y}_i A_i x.
\end{equation}

Then

\begin{equation}
\dot{\bar{\xi}} = \bar{v} r + \sum_{i=1}^{m-1} \bar{y}_i [r_0, r_1](x),
\end{equation}

\begin{equation}
\bar{\xi}(0) = 0, \quad \bar{\xi}(T) + \sum_{i=1}^{m-1} \bar{y}_i(T) A_i x_1 = 0.
\end{equation}

Hence using (7.1) we obtain

\begin{equation}
\dot{\bar{\xi}} = \bar{v} r - \sum_{i=1}^{m-1} \bar{y}_i A_i r, \quad \bar{\xi}(0) = 0.
\end{equation}

Now we can specify the right-hand boundary condition on $\bar{\xi}$. Let

$$
\mathcal{L}(x_1) = \text{Lin}(r, A_1 r, \ldots, A_{m-1} r) \cap \text{Lin}(A_1 x_1, \ldots, A_{m-1} x_1).
$$

Then from (7.17) we obtain

\begin{equation}
\bar{\xi}(T) \in \mathcal{L}(x_1).
\end{equation}

For any $\bar{\xi} \in \mathcal{L}(x_1)$ there exists $\bar{w}(\cdot) \in \mathcal{K}(x_0, x_1)$ such that $\bar{\xi}(T) = \bar{\xi}$.

Let $\bar{y} \in \mathcal{Y}(x_1)$ if there exists $\bar{\xi} \in \mathcal{L}(x_1)$ such that

\begin{equation}
\bar{\xi} + \sum_{i=1}^{m-1} \bar{y}_i A_i x_1 = 0.
\end{equation}

Since $\bar{\xi} \in \mathcal{L}(x_1)$, the set $\mathcal{Y}$ is well defined. Since $A_i x_1, i = 1, \ldots, m-1$, are elements of the basis of $\Gamma(x_1)$, the equation (7.19) has a unique solution.

Now we write down the form $\bar{\omega}(\psi, \bar{w}(\cdot))$, having in mind that $\psi = \text{const}$ on any extremal which corresponds to a trajectory from our family. By (4.3) we have

$$
\bar{\omega}(\psi, \bar{w}) = \int_0^T \sum_{i=1}^{m-1} \bar{u}_i(t) \psi A_i \bar{x} \, dt.
$$

If $\bar{w}(t) = (\bar{x}(t), \bar{u}(t)) \in \mathcal{K}(x_0, x_1)$ and $\psi \in \mathcal{G}(x_1)$, by applying the Goh transformation we obtain

\begin{equation}
\bar{\omega}(\psi, \bar{w}(\cdot)) = l(\psi, \bar{y}(T)) + \int_0^T \eta(\psi; t, \bar{y}(t)) \, dt,
\end{equation}

where

$$
l(\psi, \bar{y}(T)) = \sum_{i=1}^{m-1} \bar{y}_i(T) \psi A_i \xi(T) + \frac{1}{2} \sum_{i,k=1}^{m-1} \bar{y}_i(T) \bar{y}_k(T) \psi A_i A_k x_1,
$$

\begin{equation}
\eta = \frac{1}{2} \sum_{i,k=1}^{m-1} \bar{y}_i \bar{y}_k \psi A_i A_k r.
\end{equation}
Putting the expression for $\xi(T)$ through $\bar{y}(T)$ into $l$, we obtain

$$l(\psi, \bar{y}) = -\frac{1}{2} \sum_{i,k=1}^{m-1} \bar{y}_i \bar{y}_k A_i A_k x_1.$$  

Thus the formulas (7.20), (7.21), and (7.22) provide an expression for $\hat{\omega}$ on the cone $\mathcal{K}(x_0, x_1)$. The integral term of $\hat{\omega}$ is a form which does not depend on the trajectory of the family at all.

Let $\psi \in \text{Leg}$ if

$$\eta(\psi, t, \bar{y}) \geq 0 \ \forall \ t, \bar{y}.$$  

Let $\psi \in \text{Leg}^+$ if

$$\eta(\psi, t, \bar{y}) > 0 \ \forall \ t, \bar{y} \neq 0.$$  

By definition, $Y(x_1)$ is a subspace of $\mathbb{R}^{m-1}$ and $l$ is a quadratic form on this subspace.

Now we can turn to the investigation of rigidity. We will be interested in conditions for rigidity of any trajectory of the family that ends at the point $x_1$. We state these conditions in the following two propositions.

**Proposition 7.1.** For any trajectory of the family ending at the point $x_1$ to be rigid, it is necessary that

$$G(x_1) \cap \text{Leg} \neq \emptyset, \quad \max_{G(x_1) \cap \text{Leg}} l(\psi, \bar{y}) \geq 0 \ \forall \ \bar{y} \in Y(x_1).$$

**Proposition 7.2.** For any trajectory of the family ending at the point $x_1$ to be rigid, it is sufficient that

$$G(x_1) \cap \text{Leg}^+ \neq \emptyset, \quad \max_{G(x_1) \cap \text{Leg}} l(\psi, \bar{y}) > 0 \ \forall \ \bar{y} \in Y(x_1), \ \bar{y} \neq 0.$$  

**Proof of Proposition 7.1.** Assume the conditions (7.23) are not fulfilled. First of all, consider the case when $G(x_1) \cap \text{Leg}^+ = \emptyset$. Then the necessary condition (4.25) fails for any trajectory of the family ending at the point $x_1$. Hence any trajectory ending at $x_1$ is not rigid. This proves the necessity of the first condition (7.23).

Now suppose that the first condition is fulfilled, but the second fails. Take $\bar{y}_* \in Y(x_1)$ such that

$$\max_{G(x_1) \cap \text{Leg}} l(\psi, \bar{y}_*) < 0.$$  

Let $x_0 = x_1 - r$. Then the corresponding trajectory is defined on $[0, 1]$. Now suppose that $(\xi_*(t), \nu_*, \bar{y}_*(t)) \in \mathcal{K}(x_0, x_1)$. We will construct a sequence $(\bar{\xi}_T(t), \bar{\nu}_T, \bar{\bar{y}}_T(t))$ of elements of $\mathcal{K}(x_1 - Tr, x_1)$. Obviously, $\bar{\xi}_T(t), \bar{\bar{y}}_T(t)$ are defined on $[0, T]$. Let

$$\bar{\xi}_T(t) = \bar{\xi} \left( \frac{t}{T} \right) + \bar{\zeta}_T(t), \quad \bar{\nu}_T = \frac{1}{T} \bar{\nu}_*,$$  

$$\bar{\bar{y}}_T(t) = \frac{1}{T} \bar{\bar{y}}_* \left( \frac{t}{T} \right) + \bar{\delta}_T(t),$$

where $\bar{\delta}_T(t)$ and $\bar{\zeta}_T(t)$ are defined as follows.

Let $\Theta(\tau)$ be a scalar function continuously differentiable on $[0, 1]$ such that

$$\Theta(0) = 0, \quad \Theta(1) = 1, \quad \int_0^1 \Theta(\tau) d\tau = 0.$$
Then
\[
\tilde{\delta}_T(t) = \begin{cases} 
0, & t \in [0, T - \frac{1}{T}], \\
(1 - \frac{1}{T}) \tilde{y}_* \Theta(Tt - T^2 + 1), & t \in [T - \frac{1}{T}, T].
\end{cases}
\]
Also, let \( \tilde{\zeta}_T \) be defined by
\[
\tilde{\zeta} = -\sum_{i=1}^{m-1} \tilde{\delta}_T(t)A_i r, \quad \tilde{\zeta}_T(0) = 0.
\]
This implies that
\[
\tilde{\zeta}_T(t) = 0 \quad \text{for } t \in \left[0, T - \frac{1}{T}\right], \quad \tilde{\zeta}_T(T) = 0.
\]
Obviously,
\[
\tilde{\xi}_T = \tilde{v} r - \sum_{i=1}^{m-1} \tilde{y}_T(t)A_i r, \quad \tilde{\xi}_T(0) = 0, \quad \tilde{\xi}_T(T) = \tilde{\xi}_*(1), \quad \tilde{y}_T(T) = \tilde{y}_*.
\]
Thus
\[
\tilde{\xi}_T(t) = \tilde{v} r, \quad \tilde{y}_T(t) \in \mathcal{K}(x_1 - Tr, x_1).
\]
We have
\[
\frac{1}{2} \int_0^T \tilde{y}_T^2(t) dt \leq \int_0^T \frac{1}{T^2} \tilde{y}_*^2\left(\frac{t}{T}\right) dt + \int_0^T \tilde{\delta}_T^2(t) dt \leq \frac{1}{T} \int_0^1 \tilde{y}_*^2(\tau) d\tau + \frac{(1 - \frac{1}{T})^2}{T} \cdot \tilde{y}_*^2 \int_0^1 \Theta^2(\tau) d\tau.
\]
Therefore
\[
\int_0^T \tilde{y}_T^2(t) dt \to 0, \quad T \to +\infty.
\]
Hence we obtain
\[
\max_{G(x_1) \cap \text{Leg}} \int_0^T \eta(\psi, t, \tilde{y}_T(t)) dt \to 0, \quad T \to +\infty,
\]
which easily follows from the normalizing condition \(|\psi| = 1\).

Let \( \bar{w}_T(t) = (\bar{x}_T(t), \bar{v}_T, \bar{u}_N(t)) \), where
\[
\bar{x}_T(t) = \tilde{\xi}_T(t) + \sum_{i=1}^{m-1} \tilde{y}_T(t)A_i x(t), \quad \bar{u}_T(t) = \tilde{y}_T(t).
\]
Then we obtain \( \bar{w}_T(t) \in \mathcal{K}(x_1 - Tr, x_1) \). Putting \( \bar{w}_T(t) \) into \( \hat{\omega} \) and taking into account the representation (7.20), (7.21), (7.22), we obtain for \( \psi \in G(x_1) \)
\[
\hat{\omega}(\psi; \bar{w}_T(\cdot)) = l(\psi, \bar{y}_*) + \int_0^T \eta(\psi; t, y_T(t)) dt.
\]
Then
\[
\max_{G(x_1) \cap \text{Leg}} \hat{\omega}(\psi, \bar{w}_T(\cdot)) \leq \max_{G(x_1) \cap \text{Leg}} l(\psi, \bar{y}_*) + \max_{G(x_1) \cap \text{Leg}} \int_0^T \eta(\psi, t, y_T(t)) dt.
\]
This implies that the left-hand side is negative for sufficiently large \( T \). Therefore for sufficiently large \( T \) the necessary condition (4.7) fails.

Thus for sufficiently large \( T \) the trajectory \((x_1 - Tr, x_1)\) is not rigid. This proves the necessity of the second condition, and thus the proposition. \( \square \)
Proof of Proposition 7.2. It follows from condition (7.24) that there exists a constant \( a > 0 \) such that

\[
\max_{G(x_1) \cap \text{Leg}^+} l(\psi, \bar{y}) > a\bar{y}^2 \quad \forall \bar{y} \in Y(x_1), \quad \bar{y} \neq 0.
\]

Let \( \psi_* \in G(x_1) \cap \text{Leg}^+ \). We fix an arbitrary \( T > 0 \) and consider the trajectory \( (x_1 - Tr, x_1) \) on the interval \([0, T]\). By the choice of \( \psi_* \) we have for some \( a_1 > 0 \)

\[
(7.25) \quad \eta(\psi_*, t, \bar{y}) \geq a_1 \bar{y}^2 \quad \forall \bar{y} \in \mathbb{R}^{m-1}.
\]

Choose \( a_2 > 0 \) to satisfy the inequality

\[
l(\psi_*, \bar{y}) \geq - a_2 \bar{y}^2 \quad \forall \bar{y} \in Y(x_1).
\]

Now let

\[
\left( \xi(t), \bar{v}, \bar{y}(t) \right) \in \mathcal{K}(x_1 - Tr, x_1).
\]

Let \( \hat{\psi} \in G(x_1) \cap \text{Leg} \) be such that \( l(\hat{\psi}, \bar{y}(T)) > a_1 \bar{y}^2(T) \).

Choose \( \tau > 0 \) to satisfy the condition \( a - \tau a_2 > 0 \). Then

\[
\hat{\omega}(\hat{\psi} + \tau \psi_*, \bar{w}(\cdot)) = l(\hat{\psi} + \tau \psi_*, \bar{y}(T)) + \int_0^T \eta(\hat{\psi} + \tau \psi_*, t, \bar{y}(t)) dt,
\]

where \( \bar{w}(t) \) is the preimage of the tuple \( \hat{\xi}(t), \bar{v}, \bar{y}(t) \) under the Goh transformation.

This implies

\[
\hat{\omega}(\hat{\psi} + \tau \psi_*, \bar{w}(\cdot)) \geq \tau a_1 \int_0^T \bar{y}^2(t) dt.
\]

Since \( \bar{w}(t) \) is an arbitrary element of \( \mathcal{K}(x_1 - Tr, x_1) \), this inequality implies that the trajectory satisfies the sufficient condition (2.3). Therefore the trajectory \( (x_1 - Tr, x_1) \) is rigid. Since \( T \) is arbitrary, this proves the proposition.

Note that if \( G(x_1) \cap \text{Leg}^+ \neq \emptyset \), then, regardless of the behavior of \( \max l \), any trajectory ending at \( x_1 \) is locally rigid, which follows from the corresponding result in Section 5. Since \( G(x) \) does not change on the trajectory, when \( G(x_1) \cap \text{Leg}^+ \neq \emptyset \) the entire trajectory passing through \( x_1 \) is locally rigid.

8. STRENGTHENING OF THE NOTION OF RIGIDITY

The theory of quadratic conditions in optimal control is developed not only for a weak minimum, but also for a stronger type of minimum, namely, the Pontryagin minimum. Therefore it is natural to try to consider the degree of rigidity which would be based on the concept of Pontryagin convergence. For the differential inclusion (1.1) the Pontryagin convergence is as follows.

Let \( x^0(t) \mid [t_0, t_1] \) and \( x^k(t) \mid [t_0, t_1], \ k = 1, 2, \ldots, \) be a trajectory and a sequence of trajectories of the differential inclusion (1.1). We say that the sequence \( \{x^k(t) \mid [t_0, t_1]\} \) converges in the Pontryagin sense to the trajectory \( x^0(t) \mid [t_0, t_1] \) if and only if the following conditions are fulfilled:

\[
\max_{[t_0, t_1]} |x^0(t) - x^k(t)| \to 0, \quad \int_0^T |\dot{x}^0(t) - \dot{x}^k(t)| dt \to 0 \quad \text{as} \ k \to \infty,
\]

\[
\lim_{k \to \infty} \sup \text{ess} \sup |\dot{x}^k(t)| < \infty.
\]

However, the answer to the question of whether there exists a rigidity related to this convergence is negative. In order to see this, take a trajectory \( x^0(t) \mid [t_0, t_1] \) and construct a basis \( \hat{\tau}_0(x), \ldots, \hat{\tau}_{m-1} \) of the subspace \( \Gamma(x) \) which corresponds to this trajectory according to (2.1).
Write the differential inclusion (1.1) in this basis as the control system

\[
\dot{x} = u_0 \hat{r}_0(x) + \cdots + u_{m-1} \hat{r}_{m-1}(x).
\]

The control corresponding to the trajectory \(x^0(t) \mid [t_0, t_1]\) is \(u^0_i(t) = 1, u^0_i(t) = 0, i = 1, \ldots, m - 1\). Let \(u^k_0(t), u^k_i(t), i = 1, \ldots, m - 1, \) be the control corresponding to the trajectory \(x^k(t) \mid [t_0, t_1], k = 1, 2, \ldots\) Then it is easy to see that the Pontryagin convergence of the sequence of trajectories \(\{x^k(t) \mid [t_0, t_1]\}\) of the differential inclusion (1.1) to the trajectory \(x^0(t) \mid [t_0, t_1]\) is equivalent in the system (8.2) to the conditions

\[
\max_{[t_0, t_1]} |x^0(t) - x^k(t)| \to 0, \quad \int_{t_0}^{t_1} |1 - u^0_0(t)| \, dt \to 0 \quad \text{as} \quad k \to \infty;
\]

\[
\int_{t_0}^{t_1} |u^k_i(t)| \, dt \to 0 \quad \text{as} \quad k \to \infty, \quad i = 1, \ldots, m - 1;
\]

\[
\limsup_{k \to \infty} \sup_{t \in [t_0, t_1]} |u^k_i(t)| < +\infty, \quad i = 0, \ldots, m - 1.
\]

Now we can easily demonstrate a sequence converging to our trajectory in the Pontryagin sense, which will immediately show that the trajectory \(x^0(t) \mid [t_0, t_1]\) is not rigid.

Let \(t_*\) be an interior point of the interval \([t_0, t_1]\) and let \(\varepsilon\) satisfy the inequality \(0 < \varepsilon < t_1 - t_*\). We will construct the trajectory \((x^\varepsilon(t), u^\varepsilon_0(t), u^\varepsilon_i(t) \mid [t_0, t_1])\) of the control system (8.2) as follows. Put

\[
x^\varepsilon(t_0) = x^0(t_0),
\]

\[
u^\varepsilon_0(t) = \begin{cases} 1, & t \in [t_0, t_*], \\ 0, & t \in [t_*, t_0 + \varepsilon], \\ \frac{t_1 - t_*}{t_1 - t_* - \varepsilon}, & t \in [t_0 + \varepsilon, t_1], \end{cases}
\]

\[
u^\varepsilon_i(t) = \begin{cases} 0, & t \in [t_0, t_*], \\ 1, & t \in [t_*, t_* + \varepsilon/2], \\ -1, & t \in [t_* + \varepsilon/2, t_* + \varepsilon], \\ 0, & t \in [t_* + \varepsilon, t_1], \end{cases}
\]

\[
u^\varepsilon_i(t) = 0 \mid [t_0, t_1], \quad i = 2, \ldots, m - 1.
\]

These equalities determine the trajectory. As is easy to see, this trajectory has the following properties. On the interval \([t_0, t_*]\) the function \(x^\varepsilon(t)\) coincides with \(x^0(t)\). On the interval \([t_0, t_* + \varepsilon/2]\) it solves the equation

\[
\dot{x} = \hat{r}_1(x).
\]

On the interval \([t_* + \varepsilon/2, t_* + \varepsilon]\) it solves the equation

\[
\dot{x} = -\hat{r}_1(x).
\]

Thus at time \(t_* + \varepsilon\) the function returns to the point \(x^0(t_*)\).

On the interval \([t_* + \varepsilon, t_1]\) the function \(x^\varepsilon\) satisfies the equality

\[
x^\varepsilon(t) = x^0(\left(t - t_* - \varepsilon\right)\frac{t_1 - t_*}{t_1 - t_* - \varepsilon} + t_*).
\]

Thus \(x^\varepsilon(t_1) = x^0(t_1)\).
As $\varepsilon \to 0$, the sequence of trajectories $x^\varepsilon(t) \mid [t_0, t_1]$ converges in the Pontryagin sense to the trajectory $x^0(t) \mid [t_0, t_1]$, but on the interval $[t_*, t_* + \varepsilon]$ the function $x^\varepsilon(t)$ cannot be represented as $x^0(\varphi(t))$. Thus the trajectory $x^\varepsilon(t) \mid [t_0, t_1]$ is not rigid. But it is an arbitrary trajectory of the differential inclusion (1.1). Thus we have established that there is no Pontryagin rigidity. However, the same example indicates how we should amend the definition of the Pontryagin rigidity in order to be hopeful for success. The matter is that the Pontryagin variations allow for leaving the trajectory with subsequent returning into the same point. This has been used in the example. Therefore we can hope to get Pontryagin rigidity if we exclude the possibility of return. This is achieved by introducing the additional requirement $u^k_0(t) > c > 0$. This requirement rules out the possibility of stopping the influence of the field $\widehat{\gamma}(x)$.

Now we can formulate the notion of Pontryagin rigidity.

Let $x^0(t) \mid [t_0, t_1]$ be a solution of the differential inclusion (1.1) satisfying the smoothness requirements (1.3). Let $\widehat{\gamma}(x), \ldots, \widehat{\gamma}_{m-1}(x)$ be the basis of the distribution $\Gamma(x)$ generated by this solution.

We will say that the trajectory $x^0(t) \mid [t_0, t_1]$ is rigid in the Pontryagin sense if for any sequence

$$\{x^k(t) \mid [t_0, t_1]\}, \quad k = 1, 2, \ldots,$$

of solutions of (8.2) converging in the Pontryagin sense to $x^0(t) \mid [t_0, t_1]$ and satisfying the conditions

$$x^k(t_0) = x^0(t_0), \quad x^k(t_1) = x^0(t_1),$$

$$u^k_0(t) > c > 0 \mid [t_0, t_1], \quad k = 1, 2, \ldots,$$

there exists a sequence $\varphi^k(t)$ such that $x^k(t) = x^0(\varphi^k(t))$ for all sufficiently large $k$, where $\varphi^k(t)$ is a Lipschitz continuous monotone function defined on $[t_0, t_1]$ and satisfying the conditions $\varphi^k(t_0) = t_0, \varphi^k(t_1) = t_1$.

We will see that this rigidity can be actually realized and that it is a stronger type of rigidity than the one which has been discussed so far in this paper and treated throughout the literature on this matter.

Consider a sequence of trajectories of the differential inclusion (1.1) converging to a trajectory $x^0(t) \mid [t_0, t_1]$ and satisfying conditions (8.4).

Put

$$v^k = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} u^k_0(t) dt.$$

Then, as we know from Section 2, there exists a unique substitution $\varphi^k(t)$ such that $\psi^k(t)u^k_0(\varphi^k(t)) = v^k$.

This substitution transforms the sequence $(x^k(t), u^k_0(t), u^k(t) \mid [t_0, t_1])$ into a sequence of trajectories of the system (2.3). The conditions of Pontryagin convergence imply that $v^k \to v^0 = 1$, while $\psi^k(t)u^k_0(\varphi^k(t)) \to 0$ in $L_1$, being uniformly bounded.

Thus Pontryagin convergence in the system (8.2) combined with condition (8.4) gives rise to Pontryagin convergence in the system (2.3).

The further investigation proceeds along the same lines as in Section 2. Namely, we consider the problem (2.7) and the trajectory $x^0$. Pontryagin rigidity turns out to be equivalent to the Pontryagin maximum of the problem (2.7) at $x^0$.

Conditions for Pontryagin's extremum in problems linear in control were thoroughly studied by A. V. Dmitruk [5, 6, 7]. He established that all conditions are formulated exactly as for a weak extremum, with the only difference that instead of the set $G(\Psi_0)$ one has to consider the smaller set $G_1(\Psi_0)$. In our case it consists of elements satisfying
the additional condition
\[(8.5) \quad \psi(t) [[\hat{r}_i, \hat{r}_k], \hat{r}_s] (x^0(t)) = 0 \mid \{0, t_1\}, \quad i, k, s = 1, \ldots, m - 1.\]

In general, this condition is not invariant with respect to the choice of the basis \(\hat{r}_1(x), \ldots, \hat{r}_{m-1}(x)\). However, if two versions have the same linear span on the trajectory, condition (8.5) holds or fails simultaneously. When \(\dim \Gamma(x) = 2\), condition (8.5) is trivially fulfilled.

9. THREE EXAMPLES

In all three examples of this section \(\dim \Gamma(x) = 3\).

**Example 1.** In the space \(\mathbb{R}^4\) with elements \(x = (x_0, x_1, x_2, x_3)\) let the following linear operators \(A\) and \(B\) be given. Put \(e_0 = (1, 0, 0, 0)\), \(e_1 = (0, 1, 0, 0)\), \(e_2 = (0, 0, 1, 0)\), \(e_3 = (0, 0, 0, 1)\). Then
\[(9.1) \quad A : e_0 \mapsto e_1 \mapsto e_3 \mapsto 0, \quad e_2 \mapsto 0,\]
\[B : e_0 \mapsto e_2 \mapsto e_3 \mapsto 0, \quad e_1 \mapsto 0.\]
The distribution \(\Gamma(x)\) will be specified by the basis
\[(9.2) \quad r_0(x) = e_0, \quad r_1(x) = Ax, \quad r_2(x) = Bx.\]
We will consider the control system
\[\dot{x} = u_0r_0(x) + u_1r_1(x) + u_2r_2(x).\]
According to (9.2) it can be written as
\[(9.3) \quad \dot{x} = u_0e_0 + u_1Ax + u_2Bx.\]
We will treat this system on the set \(Q = \{x \mid \dim \Gamma(x) = 3\}\), i.e., we exclude, largely by tradition, the points where the dimension of \(\Gamma(x)\) degenerates.

According to the necessary condition (4.18), a trajectory of the system (9.3) may be considered for rigidity only if it can be completed to a Goh extremal.

Let us write down local conditions on a pair \((\psi, x)\) through which a Goh extremal can pass. Note that \(AB = BA = 0\). Therefore \([A, B] = 0\) as well. Then these conditions become
\[\psi r_0(x) = 0, \quad \psi r_1(x) = 0, \quad \psi r_2(x) = 0,\]
\[\psi [r_0, r_1](x) = 0, \quad \psi [r_0, r_2](x) = 0.\]
By (9.2) these conditions yield
\[(9.4) \quad \psi e_0 = 0, \quad \psi Ax = 0, \quad \psi Bx = 0, \quad \psi e_1 = 0, \quad \psi e_2 = 0.\]

These conditions imply that \(\psi_0 = \psi_1 = \psi_2 = 0\). Hence \(\psi = (0, 0, 0, \psi_3)\), where \(\psi_3 \neq 0\).

Then the matrix \(C = \begin{pmatrix} \psi A^2 e_0 & \psi BA e_0 \\ \psi A^2 e_0 & \psi B^2 e_0 \end{pmatrix}\) becomes \(C = \begin{pmatrix} \psi_3 & 0 \\ 0 & \psi_3 \end{pmatrix}\).

This implies that any Goh extremal is an extremal of the main stratum. According to Section 7, to these extremals there correspond trajectories with control \(u_0 = 1, \quad u_1 = u_2 = 0\). These trajectories satisfy the equations
\[(9.5) \quad \dot{x}_0 = 1, \quad \dot{x}_1 = 0, \quad \dot{x}_2 = 0, \quad \dot{x}_3 = 0.\]
Therefore, on a Goh extremal
\[x_1 = \text{const}, \quad x_2 = \text{const}, \quad x_3 = \text{const}.\]
However, it follows from (9.4) that \( x_1 = x_2 = 0 \). Indeed, otherwise
\[
\dim \text{Lin}(e_0, Ax, Bx, e_1, e_2) = 4,
\]
and no Goh extremal can pass through such a point.

Let \( x \in Q, x_1 = x_2 = 0 \). Then \( G(x) \) consists of two elements
\[
\psi^1 = (0, 0, 0, 1) \quad \text{and} \quad \psi^2 = (0, 0, 0, -1).
\]
The matrix \( C \) for \( \psi^1 \) is positive definite. Hence
\[
(9.6) \quad \psi^1 \in \text{Leg}^+.
\]
Thus we have established that in the system (9.3) rigid trajectories can be found only among solutions of equation (9.5) subject to the condition \( x_1 = x_2 = 0 \). Moreover, it follows from (9.6) that every such solution is locally rigid. We will show, however, that every trajectory satisfying (9.5) and lying in \( Q \) is rigid.

Let \( x \in \text{Lin}(e_0, e_3) \). Then
\[
(9.7) \quad Ax = x_0 e_1, \quad Bx = x_0 e_2.
\]
Hence the requirement \( \dim \Gamma(x) = 3 \) for this \( x \) is equivalent to \( x_0 \neq 0 \). Thus we will consider those trajectories satisfying (9.5) on which \( x_1 = x_2 = 0, x_0 \neq 0 \).

Let a trajectory \( \{x(0), x(T)\} \) of the family be given. Obviously, \( x_0(0) \) and \( x_0(T) \) have the same sign, and \( x_0(T) - x_0(0) = T \). For this family the basis \( r_0(x), r_1(x), r_2(x) \) has all the properties of the basis \( \hat{r}_0, \hat{r}_1, \hat{r}_2 \). Therefore in order to investigate for rigidity we have to pass to the system
\[
(9.8) \quad \dot{x} = vr_0(x) + u_1 r_1(x) + u_2 r_2(x), \quad \dot{v} = 0
\]
and consider the trajectory \( v^0 = 1, u^0_1 = u^0_2 = 0 \) with the same characteristics as before.

Let us write down the cone of critical variations \( K(x(0), x(T)) \) in terms of the variables \( \xi, \eta \).

According to (9.1) and (9.7) we obtain
\[
\begin{align*}
\dot{\xi} &= \bar{v} e_0 - \bar{y}_1(t) e_1 - \bar{y}_2(t) e_2, \quad \xi(0) = 0; \\
\bar{\xi}(T) &= -\bar{y}_1(T) Ax(T) - \bar{y}_2(T) Bx(T) = (-\bar{y}_1(T) e_1 - \bar{y}_2(T) e_2) x_0(T).
\end{align*}
\]
This immediately yields \( \bar{v} = 0 \), and hence \( \xi_0(t) = \xi_3(t) = 0 \mid [0, T] \). Then the equations can be written as
\[
(9.9) \quad \dot{\xi}_1 = -\bar{y}_1(t), \quad \dot{\xi}_2 = -\bar{y}_2(t).
\]
The boundary conditions also split up:
\[
(9.10) \quad \xi_1(0) = 0, \quad \xi_1(T) = -\bar{y}_1(T) x_0(T), \\
\xi_2(0) = 0, \quad \xi_2(T) = -\bar{y}_2(T) x_0(T).
\]
Hence it is easily seen by (7.18) that \( \mathcal{L}(x(T)) = \text{Lin}(e_1, e_2) \).

Now we can demonstrate the form \( \hat{\omega} \) for elements \( (\xi(t), \eta(t)) \in K(x(0), x(T)) \) and for \( \psi \in G(x(T)) \cap \text{Leg} \). As we have seen, the last requirement is equivalent to \( \psi = \psi^1 \). According to (7.20)–(7.22) we obtain
\[
\hat{\omega} = -\frac{1}{2} x_0(T) \left( \bar{y}_1^2(T) + \bar{y}_2^2(T) \right) + \frac{1}{2} \int_0^T \left( \bar{y}_1^2(t) + \bar{y}_2^2(t) \right) dt.
\]
By (9.10) we express \( y_1(T) \) through \( \xi_1(T) \) and \( y_2(T) \) through \( \xi_2(T) \), to obtain

\[
\hat{\omega} = -\frac{1}{2} \frac{1}{x_0(T)} \left( \xi_1^2(T) + \xi_2^2(T) \right) + \frac{1}{2} \int_0^T \left( y_1^2(t) + y_2^2(t) \right) dt.
\]

Thus the form \( \hat{\omega} \) splits into two identical forms: \( \hat{\omega} = \hat{\omega}_1 + \hat{\omega}_2 \), where

\[
\hat{\omega}_1 = -\frac{1}{2} \frac{1}{x_0(T)} \xi_1^2(T) + \frac{1}{2} \int_0^T \xi_1^2(t) dt,
\]

\[
(9.11)
\hat{\omega}_2 = -\frac{1}{2} \frac{1}{x_0(T)} \xi_2^2(T) + \frac{1}{2} \int_0^T \xi_2^2(t) dt.
\]

Now we can easily establish that the trajectory \( (x(0), x(T)) \) satisfies the sufficient conditions (5.3).

Indeed, if \( x_0(T) < 0 \) then the sufficient condition (7.24) is fulfilled. Hence by Proposition 7.2 any trajectory of the family ending at the point \( x(T) \) is rigid.

Hence it remains to consider the case \( x_0(T) > 0 \). Since the trajectory lies in \( Q \), we have \( x_0(0) > 0 \), which implies that \( x_0(T) > T \). We have \( \xi_1(T) = -\int_0^T \dot{y}_1(t) dt \). Then by the Schwarz inequality

\[
\xi_1^2(T) \leq T \int_0^T \dot{y}_1^2(t) dt.
\]

Hence

\[
\hat{\omega}_1 \geq \frac{1}{2} \left( 1 - \frac{T}{x_0(T)} \right) \int_0^T \dot{y}_1^2(t) dt.
\]

In a similar way,

\[
\hat{\omega}_2 \geq \frac{1}{2} \left( 1 - \frac{T}{x_0(T)} \right) \int_0^T \dot{y}_2^2(t) dt.
\]

Adding up, we obtain

\[
\hat{\omega} \geq \frac{1}{2} \left( 1 - \frac{T}{x_0(T)} \right) \int_0^T \left( \dot{y}_1^2(t) + \dot{y}_2^2(t) \right) dt.
\]

Hence for the trajectory \( \{x(0); x(T)\} \) the sufficient condition (5.3) is fulfilled. Thus the trajectory is rigid.

We have found all rigid trajectories of the system (9.3). Since the dimension of \( \Gamma(x) \) is as close as possible to the dimension of the space, the set of rigid trajectories is not rich. Rigid trajectories fill up the plane \( \text{Lin}(e_0, e_3) \) except for the points where \( x_0 = 0 \).

To conclude, we remark that all rigid trajectories are rigid in the Pontryagin sense as well. Indeed, the additional condition (8.5) is

\[
\psi[r_1, r_2]r_1 = 0, \quad \psi[r_1, r_2]r_2 = 0.
\]

But this condition is trivially fulfilled, since \([A, B] = 0\).

**Example 2.** In this example \( n = 5 \). We will write the elements of \( \mathbb{R}^5 \) as \( x = (x_0, x_1, x_2, x_3, x_4) \). Let \( e_i = (\delta_{i0}, \ldots, \delta_{i4}) \), \( i = 0, 1, \ldots, 4 \). Define the operators

\[
A : e_0 \mapsto e_1 \mapsto e_4 \mapsto 0, \quad e_2 \mapsto 0, \quad e_3 \mapsto e_3,
\]

\[
B : e_0 \mapsto e_2 \mapsto e_4 \mapsto 0, \quad e_1 \mapsto 0, \quad e_3 \mapsto e_3.
\]

Put \( r_0(x) = e_0, \ r_1(x) = Ax, \ r_2(x) = Bx \) and consider the three-dimensional distribution \( \Gamma(x) = \text{Lin}(r_0(x), r_1(x), r_2(x)) \). It can be easily verified that

\[
[A, B] = 0, \quad ABr_0 = BAr_0 = 0,
\]

\[
A^2r_0 = e_4, \quad B^2r_0 = e_4.
\]

(9.12)
Consider the control system

\begin{equation}
\dot{x} = u_0 e_0 + u_1 Ax + u_2 Bx.
\end{equation}

We have \( x_3 = (u_1 + u_2)x_3 \). Hence the subspace of \( \mathbb{R}^5 \) for which \( x_3 = 0 \) is invariant for the system (9.13). On this subspace this example coincides with the example considered above. Therefore it is of interest to consider the set \( x_3 \neq 0 \), which is also invariant with respect to the system (9.13).

This example is of the type considered in Section 7. According to (9.12) a Goh extremal of the system (9.13) must satisfy the conditions

\begin{equation}
\psi_0 = 0, \quad \psi Ax = 0, \quad \psi Bx = 0, \quad \psi e_1 = 0, \quad \psi e_2 = 0.
\end{equation}

This implies that the following condition must be fulfilled:

\[ \dim \text{Lin}(e_0, Ax, Bx, e_1, e_2) < 5. \]

By assumption, \( x_3 \neq 0 \), hence \( Ax \notin \text{Lin}(e_0, e_1, e_2) \). Thus the above inequality can be rewritten for as

\begin{equation}
\dim \text{Lin}(e_0, Ax, Bx, e_1, e_2) = 4.
\end{equation}

We have

\[ Ax = x_0 e_1 + x_1 e_4 + x_3 e_3, \quad Bx = x_0 e_2 + x_2 e_4 + x_3 e_3. \]

Therefore (9.15) is fulfilled if and only if

\begin{equation}
x_1 = x_2.
\end{equation}

Then condition \( \dim \Gamma(x) = 3 \) implies

\begin{equation}
x_0 \neq 0.
\end{equation}

So, let \( x_1 = x_2 = \theta \). It is easily seen that conditions (9.14) imply

\[ \psi_0 = \psi_1 = \psi_2 = 0, \quad \psi_3 x_3 + \psi_4 \theta = 0, \]

whence

\begin{equation}
\psi_4 = \mu x_3, \quad \psi_3 = -\mu \theta, \quad \mu \neq 0.
\end{equation}

The matrix

\[ C = \begin{pmatrix} \psi A^2 e_0 & \psi B A e_0 \\ \psi A B e_0 & \psi B^2 e_0 \end{pmatrix} \]

becomes

\begin{equation}
C = \begin{pmatrix} \mu x_3 & 0 \\ 0 & \mu x_3 \end{pmatrix},
\end{equation}

with \( \mu x_3 \neq 0 \). This implies that the pair \( (\psi, x) \) satisfies the conditions which determine an extremal of the main stratum. Then, according to Section 7, we should seek for rigid extremals only among solutions of the system for which \( u_0 = 1, u_1 = u_2 = 0 \).

Then

\begin{equation}
x_0 = x_0(0) + t, \quad x_1 = x_2 = \theta = \text{const},
\end{equation}

\[ x_3 = \text{const}, \quad x_4 = \text{const}. \]

The condition on the trajectory is that

\begin{equation}
x_0(0) \neq 0, \quad x_0(0) \text{ and } x_0(T) \text{ have the same sign.}
\end{equation}

Consider a trajectory joining the points \( x(0) \) and \( x(T) \) and satisfying conditions (9.20) and (9.21).
With respect to this family the basis $e_0, Ax, Bx$ has the properties of the basis $\hat{r}_0, \hat{r}_1, \hat{r}_2$ (see (2.1)). Hence we can pass to the system
\begin{equation}
\dot{x} = ve_0 + u_1 Ax + u_2 Bx, \quad \dot{v} = 0
\end{equation}
and consider the family in the framework of this system.

Let us write down the cone $K(x(0), x(T))$ for the trajectory $(x(0), x(T))$ in the variables $\xi, \eta$. By (7.17) we obtain
\begin{equation}
\dot{\xi} = \tilde{v}e_0 - \tilde{y}_1(t)e_1 - \tilde{y}_2(t)e_2, \quad \xi(0) = 0.
\end{equation}
The right-hand boundary condition has the form

$$
\tilde{\xi}(T) = - \tilde{y}_1(T) Ax(T) - \tilde{y}_2(T) Bx(T).
$$

Since $Ax, Bx \in \text{Lin}(e_1, e_2, e_3, e_4)$ for any $x$, we have $\tilde{v} = 0$, whence $\tilde{\xi}_0(t) = 0$ on $[0, T]$. Furthermore,
\begin{align*}
\dot{\xi}_1 &= \tilde{y}_1(t), \quad \dot{\xi}_2 = - \tilde{y}_2(t), \quad \tilde{\xi}(t) = \text{const} | [0, T], \quad \xi_4 = \text{const} | [0, T].
\end{align*}
Hence, by the left-hand boundary condition, $\tilde{\xi}_3(t) = \tilde{\xi}_4(t) = 0 | [0, T]$. But
\begin{equation}
Ax(T) = x_0(T)e_1 + \theta e_4 + x_3 e_3, \quad Bx(T) = x_0(T)e_2 + \theta e_4 + x_3 e_3.
\end{equation}

Therefore
\begin{equation}
\theta (\tilde{y}_1(T) + \tilde{y}_2(T)) = 0, \quad x_3 (\tilde{y}_1(T) + \tilde{y}_2(T)) = 0.
\end{equation}
Since $x_3 \neq 0$, we have
\begin{equation}
\tilde{y}_1(T) + \tilde{y}_2(T) = 0.
\end{equation}

Since, according to the right-hand boundary condition,
\begin{equation}
\tilde{\xi}_1(T) = - \tilde{y}_1(T)x_0(T), \quad \tilde{\xi}_2(T) = - \tilde{y}_2(T)x_0(T),
\end{equation}
we obtain from (9.25)
\begin{equation}
\tilde{\xi}_1(T) + \tilde{\xi}_2(T) = 0.
\end{equation}
It is seen from (9.27) that in this example the cone $K$ has a structure different from that in the previous example, where there were no restrictions on $\xi_1(T)$ and $\xi_2(T)$.

Let us normalize the extremals corresponding to the trajectory $\{x(0); x(T)\}$, more precisely, to the point $x(T)$, by the condition $|\mu| = 1$. Then by (9.18) $G(x(T)) = \{\psi^1; \psi^2\}$, where
\begin{equation}
\psi^1 = (0, 0, 0, -\theta, x_3), \quad \psi^2 = (0, 0, 0, \theta, -x_3).
\end{equation}
The matrix $C$ corresponding to the one of these two elements for which $\mu x_3 > 0$ is positive definite, hence $\psi \in \text{Leg}^+$. According to Section 5, this implies that any trajectory of the family at hand is locally rigid.

Let us write down the form $\tilde{\omega}$ in variables $\xi, \eta$ for $\psi \in G(x(T)) \cap \text{Leg}$ and for an element of the cone $K(x(0), x(T))$. By (7.20)-(7.22) we have
\begin{equation}
\tilde{\omega} = l(\psi, \eta(T)) + \frac{1}{2} \int_0^T |x_3| \cdot (\tilde{y}_1^2(t) + \tilde{y}_2^2(t)) dt,
\end{equation}
where
\begin{equation}
l(\psi, \eta) = -\frac{1}{2} \tilde{y}_1^2 \psi A^2 x(T) - \frac{1}{2} \tilde{y}_1 \tilde{y}_2 (\psi AB \tilde{x}(T) + \psi BA \tilde{x}(T)) - \frac{1}{2} \tilde{y}_2^2 \psi B^2 x(T).
\end{equation}
RIGIDITY AND OPTIMAL CONTROL

But

\[ A^2x(T) = x_0(T)e_4 + x_3e_3, \]
\[ ABx(T) = BAx(T) = x_3e_3, \]
\[ B^2x(T) = x_0(T)e_4 + x_3e_3. \]

Hence

\[ \psi A^2x(T) = \mu x_0(T)x_3 - \mu \theta x_3, \]
\[ \psi ABx(T) = \psi BAx(T)x_3 = -\mu \theta x_3, \]
\[ \psi B^2x(T) = \mu x_0(T)x_3 - \mu \theta x_3. \]

Putting these expressions into \( l \), we obtain

\[ (9.30) \quad l = -\frac{1}{2} \left( \tilde{y}_1^2 (\mu x_0(T)x_3 - \mu \theta x_3) + \tilde{y}_1 \tilde{y}_2 (-2\mu \theta x_3) + \tilde{y}_2^2 (\mu x_0(T)x_3 - \mu \theta x_3) \right). \]

Here \( \mu = x_3/|x_3| \). But, by \( (9.25) \), \( \tilde{y}(T) \in Y \), where \( Y = \{ \tilde{y} \mid \tilde{y}_1 + \tilde{y}_2 = 0 \} \). Hence the form \( l \) on \( Y \) becomes

\[ (9.31) \quad l = -\frac{1}{2} \left( \tilde{y}_1^2 + \tilde{y}_2^2 \right) |x_3|x_0(T). \]

The further treatment proceeds as in the previous example.

As a result, we have that each trajectory of our family \( (9.20), (9.21) \) is rigid, specifically, rigid in the Pontryagin sense because \([A, B] = 0\).

We see that a slight complication of the operators \( A \) and \( B \) sharply increased the dimension of the set filled by rigid trajectories: it leapt up from two in the previous example to four. As in the previous example, we have found all rigid trajectories of the control system.

**Example 3.** Consider the last example in this series. Let \( n = 6 \). Then \( x = (x_0, \ldots, x_5) \).

Let \( e_i = (\delta_{i0}, \ldots, \delta_{i5}), \ i = 0, \ldots, 5 \). Define two operators in \( \mathbb{R}^6 \):

\[ A : e_0 \mapsto e_1 \mapsto e_5 \mapsto 0; \quad e_2 \mapsto 0; \quad e_3 \mapsto e_1 + e_3; \quad e_4 \mapsto e_1 + e_3; \]

\[ B : e_0 \mapsto e_2 \mapsto e_5 \mapsto 0; \quad e_1 \mapsto 0; \quad e_3 \mapsto e_2 + e_4; \quad e_4 \mapsto e_2 + e_4. \]

Let \( \Gamma(x) = \text{Lin}(e_0, Ax, Bx) \). We are interested in the set where \( \dim \Gamma(x) = 3 \). We have

\[ BA : e_0 \mapsto 0, \quad e_1 \mapsto 0, \quad e_2 \mapsto 0, \quad e_5 \mapsto 0, \]
\[ e_3 \mapsto e_2 + e_4; \quad e_4 \mapsto e_2 + e_4; \]
\[ AB : e_0 \mapsto 0, \quad e_1 \mapsto 0, \quad e_2 \mapsto 0, \quad e_5 \mapsto 0, \]
\[ e_3 \mapsto e_1 + e_3; \quad e_4 \mapsto e_1 + e_3. \]

Consider the control system

\[ (9.32) \quad \dot{x} = u_0e_0 + u_1 Ax + u_2 Bx. \]

Clearly, \( AB e_0 = BA e_0 \). Hence the system \( (9.32) \) is of the type considered in Section 7.

According to \( (7.11), (7.12) \) the local conditions to be satisfied by a Goh extremal are

\[ (9.33) \quad \psi e_0 = 0, \quad \psi Ax = 0, \quad \psi Bx = 0, \]
\[ \psi e_1 = 0, \quad e_2 = 0, \quad \psi[A, B]x = 0. \]

This implies that

\[ (9.34) \quad \dim \text{Lin}(e_0, e_1, e_2, Ax, Bx, [A, B]x) < 6. \]
We have
\begin{equation}
Ax = x_0 e_1 + x_1 e_5 + (x_3 + x_4)e_1 + (x_3 + x_4)e_3, \\
Bx = x_0 e_2 + x_2 e_5 + (x_3 + x_4)e_2 + (x_3 + x_4)e_4, \\
[A, B]x = (x_3 + x_4)(e_1 + e_3) - (x_3 + x_4)(e_2 + e_4).
\end{equation}
(9.35)

Consider the two cases: 1) \(x_3 + x_4 \neq 0\) and 2) \(x_3 + x_4 = 0\). In the first case
\begin{equation}
x_1 = x_2 = 0,
\end{equation}
(9.36)
otherwise condition (9.34) fails. Once (9.36) holds, we have
\[ \dim \text{Lin}(e_0, e_1, e_2, Ax, Bx, [A, B]x) = 5. \]

In the second case
\[ \dim \text{Lin}(e_0, e_1, e_2, Ax, Bx, [A, B]x) \leq 4. \]

In the first case the condition \(\dim \Gamma(x) = 3\) was fulfilled automatically. In the second case the inequality \(x_0 \neq 0\) must be fulfilled. Note that
\begin{equation}
\dot{x}_3 = u_1(x_3 + x_4), \quad \dot{x}_4 = u_2(x_3 + x_4).
\end{equation}
(9.37)
This implies that the sets \(x_3 + x_4 \neq 0\) and \(x_3 + x_4 = 0\) are invariant with respect to the control system (9.32).

First we consider the case
\begin{equation}
x_3 + x_4 \neq 0, \quad x_1 = x_2 = 0.
\end{equation}
(9.38)
In this case condition (9.33) is equivalent to
\begin{equation}
\psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0, \quad \psi_5 \neq 0.
\end{equation}
(9.39)
Since \(A^2 e_0 = B^2 e_0 = e_5\) and \(ABe_0 = BAe_0 = 0\), the matrix \(C\) has the form
\[ C = \begin{pmatrix} \psi_5 & 0 \\ 0 & \psi_5 \end{pmatrix}. \]

According to Section 7 this implies that any extremal satisfying the Goh conditions is an extremal of the main stratum. Hence
\[ u_0 = 1, \quad u_1 = u_2 = 0. \]

In other words, rigid trajectories can be found only in the family \(\dot{x} = e_0\). On these trajectories
\[ x_1, x_2, x_3, x_4, x_5 = \text{const}. \]
Moreover, by (9.38), for these trajectories \(x_1 = x_2 = 0, x_3 + x_4 \neq 0\). With respect to this family the basis \(e_0, Ax, Bx\) has all the properties of the basis \(\hat{r}_0, \hat{r}_1, \hat{r}_2\) (see (2.1)). Hence the conditions for rigidity are stated in terms of the system
\begin{equation}
\dot{x} = ve_0 + u_1 Ax + u_2 Bx, \quad \dot{v} = 0.
\end{equation}
(9.40)

Let a trajectory of this family joining the points \(x(0)\) and \(x(T)\) satisfy (9.38). By (9.39) the set \(G(x(T))\) consists of two points: \(\psi^1 = e_5, \psi^2 = -e_5\). Obviously, \(\psi^1 \in \text{Leg}^+\). This implies that any trajectory of this family is locally rigid.

Let us continue our investigation and find \(K(x(0), x(T))\). According to Section 7 we obtain
\begin{equation}
(\xi(t), \bar{y}(t)) \in K(x(0), x(T)) \iff \dot{\xi} = \bar{v} e_0 - \bar{y}_1(t)e_1 - \bar{y}_2(t)e_2, \quad \xi(0) = 0.
\end{equation}
(9.41)
The right-hand boundary condition has the form

\[ \xi(T) = -y_1(T)Ax(T) - y_2(T)Bx(T). \]

Multiplying by \( e_3 \), we obtain

\[ \dot{\xi}_3(T) = y_1(T)Ax(T)e_3 + y_2(T)Bx(T)e_3 = y_1(T)(x_3 + x_1). \]

But (9.41) implies that \( \overline{v} = 0, \xi_3(t) = 0 \text{ \forall } [0,T]. \) Therefore \( y_1(T) = 0. \) In a similar way, multiplying (9.41) by \( e_4 \) we obtain \( y_2(T) = 0. \) Hence \( \xi(T) = 0. \) Then for \( \psi \in G(x(T) \cap \text{Leg}) \) and for \( (\xi, \dot{y}) \in K(x(0), x(T)) \) we have

\[ \hat{\omega} = \frac{1}{2} \int_0^T (y_1^2(t) + y_2^2(t)) \, dt. \]

Thus for any trajectory of the family the sufficient condition for rigidity is fulfilled. Therefore any trajectory of the family is rigid.

Let us find out whether they are rigid in the Pontryagin sense. By (8.5) the additional necessary condition in this case has the form

\[ \psi [[A, B], A] x = 0, \quad \psi [[A, B], B] x = 0. \]

Since \( \psi^1 = e_5 \), this implies that

\[ \{ [[A, B], A] x, \quad [[A, B], B] x \} \in \text{Lin}(e_0, e_1, e_2, e_3, e_4). \]

But this requirement is not fulfilled if \( x_3 + x_4 \neq 0. \) Indeed, it is easily verified that

\[ [A, B]x = (x_3 + x_4)e_1 - (x_3 + x_4)e_2 + (x_3 + x_4)e_3 - (x_3 + x_4)e_4. \]

Hence

\[ A[A, B]x = (x_3 + x_4)e_5, \quad B[A, B]x = (x_3 + x_4)e_5. \]

On the other hand, it is easy to verify that

\[ e_5[A, B]Ax = e_5[A, B]Bx = 0, \quad \forall \ x \in \mathbb{R}^6. \]

This implies that condition (8.5) on the trajectories of the family fails; hence no trajectory of the family is rigid in the Pontryagin sense.

Thus we have completely investigated the first case.

In the second case, according to (9.37), the subspace \( x_3 + x_4 = 0 \) stratifies into layers invariant with respect to the control system, which are characterized by the condition

\[ x_3 - x_4 = \text{const}. \]

Hence it is natural to consider the problem of rigidity separately in each layer. But one can see that in each layer the system (9.32) turns into the system of the first example. Thus the investigation of the second case is complete.

This also completes the investigation of the system (9.32). We have found all rigid trajectories and established that they are split into two families. In the first, richer, family no trajectory is rigid in the Pontryagin sense; in the second family all trajectories possess Pontryagin rigidity.

Thus the notion of Pontryagin rigidity is really meaningful: it strengthens the notion of rigidity and does not reduce to it.

Apparently, there exists a strong rigidity as well, i.e., a rigidity which corresponds to the strong extremum in the problem (2.7), but optimal control theory does not have enough results on strong minima in problems without local constraints and linear in control.
10. CONCLUSION

In this section we tackle some relevant issues which were not covered in the main text of the paper.

1. It appears to us that an important issue would be the reduction of rigidity to a notion which could be called rigid association. It consists in the following. Suppose we are given a distribution $\Gamma(x)$ and the corresponding differential inclusion

\begin{equation}
\dot{x} \in \Gamma(x).
\end{equation}

Let $D(x)$ be a univariate distribution such that $D(x) \subseteq \Gamma(x)$ for any $x$. We will say that a trajectory $x(t) \mid [t_0, t_1]$ of the differential inclusion

\begin{equation}
\dot{x} \in D(x)
\end{equation}

is rigidly associated with the differential inclusion (10.2) if any trajectory of the differential inclusion (10.1) defined on $[t_0, t_1]$, if it sufficiently close to the trajectory $x(t) \mid [t_0, t_1]$ and has the same boundary values, is a trajectory of the differential inclusion (10.2).

This is the notion of rigid association which reduces to the notion of extremum.

2. With each trajectory investigated for rigidity we associated a univariate distribution $D(x)$ by choosing a special basis $\tilde{r}_0(x), \ldots, \tilde{r}_{m-1}(x)$. This basis was constructed under the assumptions (1.3), which required the trajectory to be specified by a thrice continuously differentiable function with derivative, which nowhere vanishes, and to have no points of self-crossing.

While the first two requirements are essential, the last was introduced only for convenience of presentation. All conditions obtained in Sections 3–5 are not connected with its fulfillment. This can be seen as follows.

Let $x^0(t) \mid [t_0, t_1]$ be a trajectory of the differential inclusion (10.1) which satisfies the first two requirements but not the third.

Take an arbitrary function $x^1(t) \mid [t_0, t_1]$ satisfying all three requirements (1.3). Then there exists a thrice continuously differentiable mapping $f$ taking some neighborhood $Q^1$ of the set $x^1([t_0, t_1])$ to some neighborhood $Q^0$ of the set $x^0([t_0, t_1])$ and satisfying the conditions

\begin{equation}
\det f'(x) \neq 0, \quad \forall x \in Q^1, \quad f(x^1(t)) = x^0(t), \quad \forall t \in [t_0, t_1].
\end{equation}

For $x \in Q^1$ let $\Gamma^1(x) = f'(x)^{-1}\Gamma(f(x))$, and consider the differential inclusion

\begin{equation}
\dot{x} \in \Gamma^1(x)
\end{equation}

defined on $Q^1$. It is clear that $x^1(t) \mid [t_0, t_1]$ is a trajectory of the differential inclusion (10.3).

One can see that the trajectories $x^0(t) \mid [t_0, t_1]$ and $x^1(t) \mid [t_0, t_1]$ are equivalent with regard to rigidity. Hence we can obtain conditions on $x^0(t) \mid [t_0, t_1]$ by applying the conditions of Sections 3–5 to $x^1(t) \mid [t_0, t_1]$ and then restating them in terms of $x^0(t) \mid [t_0, t_1]$. One can see that they do not differ from the ones obtained before.

Thus rigidity is reduced to rigid association in a fairly general setup.

3. Instead of the differential inclusion (10.1) we investigated an equivalent control system, which was specified by a basis in $\Gamma(x)$.

There are different types of control system equivalent to the differential inclusion (10.1), but the one we have used is most suitable for the derivation of conditions for rigidity as well as for their subsequent application. In support of this we mention the following two points.

First, the "basis" control system is closely related to a family of extremals important for the theory of rigidity, namely, the extremals of the main stratum. Second, using the
basis representation of the differential inclusion (10.1) we can construct interesting and nontrivial examples and thoroughly investigate them for rigidity.

4. In Section 6 we pointed out that the sufficient conditions for rigidity which we obtained did not assume that the bracket powers of $\Gamma$ retain a constant dimension in a neighborhood of the trajectory under consideration. Note that the examples considered demonstrate that this strengthening is not a mere formality, since in all examples, except for Example 1 in Section 6, the dimension of the bracket powers of $\Gamma$ was not constant in any neighborhood of the trajectory under investigation. At the same time each trajectory considered is rigid on some set of time intervals, which was established by means of the sufficient conditions we obtained.

5. In Section 4 we presented the cone of critical variations $\mathcal{K}$ and the quadratic form $\omega$ for a Goh element $\Psi_0$ in a form which results from the Goh transformation. However this is not the final form in the theory of quadratic conditions. In the final form $\tilde{y}(t)$ is an arbitrary bounded measurable function, and hence the value of $\tilde{y}(t_1)$ is not connected with it. We will state this form as well as the complete expression for the quadratic form $\omega$.

The set $\mathcal{K}$ consists of triplets

\begin{equation}
(10.4) \quad (\xi(t), \tilde{y}(t), \tilde{\beta} | \tilde{y}(t) \in L_{\infty}, \tilde{\beta} \in \mathbb{R}^{m-1}).
\end{equation}

A triplet $(\xi(\cdot), \tilde{y}(\cdot), \tilde{\beta})$ belongs to $\mathcal{K}$ if and only if

\begin{equation}
(10.5) \quad \dot{\xi} = \tilde{\mathcal{R}}_0(x^0(t))\xi + \tilde{\nu}(x^0(t)) + \sum_{i=1}^{m-1} \tilde{y}_i(t)[\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_i](x^0(t)),
\end{equation}

\begin{equation}
(10.6) \quad \xi(t_0) = 0, \quad \tilde{\xi}(t_1) = - \sum_{i=1}^{m-1} \tilde{\beta}_i \tilde{\mathcal{R}}_i(x^0(t_1)).
\end{equation}

For a Goh element of $\Psi_0$ the quadratic form $\omega$ can be written as $\omega = \omega_1 + \omega_2$, where

\begin{equation}
(10.7) \quad \omega_1 = -\frac{1}{2} \sum_{i,k=1}^{m-1} \tilde{\beta}_i \tilde{\beta}_k a_{ik},
\end{equation}

with $a_{ik} = \psi(t_1)\tilde{\mathcal{R}}_i'(x^0(t_1))\tilde{\mathcal{R}}_k(x^0(t_1))$, $i, k = 1, \ldots, m - 1$, and

\begin{equation}
(10.8) \quad \omega_2 = \int_{t_0}^{t_1} \left( \frac{1}{2} \psi(t)\tilde{\mathcal{R}}_0^2(x^0(t))\dot{\xi}(t) + \tilde{\nu}(x^0(t))\tilde{\mathcal{R}}_0(x^0(t))\dot{\xi}(t) 
\right.

+ \sum_{i=1}^{m-1} \tilde{y}_i(t)\psi(t)[\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_i](x^0(t))\dot{\xi}(t) + \frac{1}{2} \sum_{i,k=1}^{m-1} \tilde{y}_i(t)\tilde{y}_k(t)b_{ik}(t) \right) dt,
\end{equation}

with $b_{ik}(t) = \psi(t) [\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_i, \tilde{\mathcal{R}}_k](x^0(t))$, $i, k = 1, \ldots, m - 1$.

The formulations of conditions for rigidity remain the same as in Sections 4 and 5.

6. The space $\mathbb{R}^3$ provides great opportunities for constructing interesting examples. Let $x = (x_0, x_1, x_2)$, $\Gamma(x) = \text{Lin}(r_0(x), r_1(x))$. Put $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$. We will consider $r_0(x)$ and $r_1(x)$ of the form $r_0(x) = e_0$, $r_1(x) = f(x)e_1 + e_2$. One can see that this form does not restrict generality.

Write down the differential inclusion (1.1) as the control system

\begin{equation}
(10.9) \quad \dot{x} = u_0 r_0(x) + u_1 r_1(x).
\end{equation}

Let us describe the Goh extremals of the system (10.9). Put

\begin{equation}
\begin{align*}
c(x) &= [r_1, r_0](x), \quad p(x) = [c, r_0](x), \quad q(x) = [c, r_1](x).
\end{align*}
\end{equation}
Then, obviously, \( c = \varphi(x)e_1 \), where \( \varphi(x) = f_{x_0}'(x) \), and

\[
(10.10) \quad p = \varphi'_{x_0}(x)e_1, \quad q(x) = (\varphi'_{x_1}(x)f(x) + \varphi'_{x_2}(x) - f'_{x_1}(x)\varphi(x))e_1.
\]

Set \( \tilde{\psi}(x) = e_1 - f(x)e_2 \). Obviously, \( \tilde{\psi}(x) \perp \Gamma(x) \) for any \( x \). Since \( r_0(x), r_1(x), \tilde{\psi}(x) \) form a basis of \( \mathbb{R}^3 \),

\[
\psi \Gamma(x) = 0 \implies \psi \parallel \tilde{\psi}(x).
\]

This implies that the points through which a Goh extremal can pass must satisfy the condition \( \tilde{\psi}(x)c(x) = 0 \), which is equivalent to the condition \( \varphi(x) = 0 \). Denote by \( M_0 \) the set of points \( x \) for which \( \varphi(x) = 0 \).

Put

\[
\mathcal{D}(x) = \left\{ u \mid u_0 (\tilde{\psi}(x), p(x)) + u_1 (\tilde{\psi}(x), q(x)) = 0 \right\}
\]

and consider the control system

\[
(10.11) \quad \dot{x} = u_0 r_0(x) + u_1 r_1(x), \quad u \in \mathcal{D}(x).
\]

It is clear that a trajectory corresponding to a Goh extremal solves the system (10.11) and its state component belongs to \( M_0 \). The converse statement is also true.

**Proposition 10.1.** Let \( x(t), u(t) \mid [t_0, t_1] \) be a solution of the system (10.11) and \( x(t_0) \in M_0 \). Then \( x(t), u(t) \mid [t_0, t_1] \) is the state component of a Goh extremal.

**Proof.** First we prove that \( x(t) \in M_0 \mid [t_0, t_1] \), i.e., that \( \varphi(x(t)) = 0 \mid [t_0, t_1] \). We have

\[
\frac{d\varphi}{dt} = \varphi'_{x_0}(x(t))u_0(t) + (\varphi'_{x_1}(x(t))f(x(t)) + \varphi'_{x_2}(x(t)))u_1(t).
\]

By (10.10) this implies

\[
\frac{d\varphi}{dt} = u_0(t)\tilde{\psi}(x(t))p(x(t)) + u_1(t)\tilde{\psi}(x(t))q(x(t)) + f'_{x_1}(x(t))\varphi(x(t)).
\]

Since the trajectory \( x(t), u(t) \mid [t_0, t_1] \) solves the system (10.11), we obtain

\[
\frac{d\varphi}{dt} = f'_{x_1}(x(t))\varphi(x(t)).
\]

But, by assumption, \( \varphi(x(t_0)) = 0 \). Therefore \( \varphi(x(t)) = 0 \mid [t_0, t_1] \). Thus we have proved that \( x(t) \mid [t_0, t_1] \) belongs to \( M_0 \).

We continue the proof. The adjoint system for (10.9) has the form

\[
(10.12) \quad - \frac{d\psi_0}{dt} = u_1(t)\psi_1\varphi(x(t)),
\]

\[
- \frac{d\psi_1}{dt} = u_1(t)\psi_1 f'_{x_1}(x(t)),
\]

\[
- \frac{d\psi_2}{dt} = u_1(t)\psi_1 f'_{x_2}(x(t)).
\]

Putting \( \psi_2 = \zeta \psi_1 \), rewrite the system (10.12) as

\[
- \frac{d\psi_0}{dt} = u_1(t)\psi_1\varphi(x(t)),
\]

\[
- \frac{d\psi_1}{dt} = u_1(t)\psi_1 f'_{x_1}(x(t)),
\]

\[
- \frac{d\zeta}{dt} = u_1(t)\left(-\zeta f'_{x_1}(x(t)) + f'_{x_2}(x(t))\right).
\]
Putting $\psi(t_0) = \hat{\psi}(x(t_0))$ and taking the equality $\varphi(x(t)) = 0 \mid [t_0, t_1]$ into account, we obtain $\psi_0(t) = 0 \mid [t_0, t_1]$ and $\zeta(t) = -f(x(t)) \mid [t_0, t_1]$. Then $\psi(t)\|\hat{\psi}(x(t))$. Thus the proposition is proved. □

Define $\hat{u}(x)$ by setting
\[
\hat{u}_0(x) = -\hat{\psi}(x)q(x), \quad \hat{u}_1(x) = \hat{\psi}(x)p(x).
\]
Obviously, $\hat{u}(x) \in D(x)$, with $D(x)$ one-dimensional if $\hat{u}(x) \neq 0$, and $D(x)$ equal to $\mathbb{R}^2$ if $\hat{u}(x) = 0$. Consider the equation
\[
(10.13) \quad \dot{x} = \hat{u}_0(x)r_0(x) + \hat{u}_1(x)r_1(x).
\]
It follows from Proposition 10.1 and (10.13) that through each point $x \in M_0$ such that $\hat{u}(x) \neq 0$ there passes a single Goh extremal, which necessarily belongs to the main stratum.

We will call a point $x \in M_0$ such that $\hat{u}(x) = 0$ a singular point. These and only these are the points through which no extremal of the main stratum can pass. It follows from the aforesaid, in particular, that if an extremal has break points then each such point is singular.

7. Consider the class of distributions where $\varphi$ is a function of $x_0$ and $x_2$. Then
\[
(10.14) \quad \hat{u}_0(x) = -\varphi'_{x_2}, \quad \hat{u}_1(x) = \varphi'_{x_0}.
\]
The system (10.13) becomes
\[
(10.15) \quad \dot{x}_0 = -\varphi'_{x_2}, \quad \dot{x}_2 = \varphi'_{x_0}.
\]
The singular points of the system (10.15) are those where $\text{grad} \varphi = 0$, $\varphi = 0$. Particular cases of (10.15) provide very interesting examples.

Let
\[
(10.16) \quad \varphi(x_0, x_2) = x_0 x_2, \quad f(x_0, x_2) = \frac{1}{2} x_0^2 x_2.
\]
In this case we can find all rigid trajectories.

The system (10.15) has the form
\[
\dot{x}_0 = -x_0, \quad \dot{x}_2 = x_2.
\]
On each Goh extremal the variables $x_0, x_2$ belong to the coordinate axes. The origin is a singular point. At a passage through the origin, the transition from one coordinate axis to the other is possible. We will consider only two families of trajectories. This will suffice for getting an idea of rigid trajectories in this case.

The first family is described by the conditions
\[
u_0(t) = 1, \quad u_1(t) = 0, \quad x_2(t) = 0.
\]
Since the system (10.9) is invariant with respect to translations of the independent variable, we obtain $x_0(0) = 0$. Obviously, $f = 0$ on a trajectory of this type; hence $x_1(t) = \text{const} = c$. The family is parametrized by the time interval and $c$.

The second family of trajectories is defined by the following conditions: the time interval is $[-T, T]$,
\[
\begin{align*}
u_0(t) &= 1, \quad u_1(t) = 0, \quad x_2(t) = 0 & \text{on } [-T, 0], \\
u_0(t) &= 0, \quad u_1(t) = 1, \quad x_0(t) = 0 & \text{on } [0, T].
\end{align*}
\]
Obviously, $x_0(0) = x_2(0) = 0$. Each trajectory of the second family is not smooth. Since $f = 0$ on each trajectory, one has $x_1(t) = \text{const} = \tilde{x}_1$ on the trajectory. Thus the trajectories of the second family are parametrized by $T$ and $\tilde{x}_1$.

We will show that each trajectory of the second family is rigid. This will imply that any trajectory of the first family whose time interval contains no singular point in its interior is rigid. We will show that whenever there is a singular point inside the time interval, a trajectory of the first family is not rigid. This can easily be done using the conditions of Section 4. Indeed, either the Legendre term has everywhere the same sign as $\tilde{\psi}(x(t))q(x(t))$, or it has everywhere the opposite sign. But $\tilde{\psi}(x)q(x) = x_0(t)$, and on each trajectory of the first family $x_0(t) = t$. To the singular point there corresponds the point $t = 0$. Thus the Legendre coefficient changes its sign at $t = 0$; hence $\text{Leg}(\Psi_0) = \emptyset$ for the trajectory. According to Section 4 this implies that the trajectory is not rigid.

Now we turn to the trajectories of the second family. Since all of them are not smooth, we cannot apply the conditions of Section 5. We will establish their rigidity directly.

So, fix some $T$ and $\tilde{x}_1$ and consider the corresponding trajectory. Take a trajectory of the system (10.9) which is defined on $[-T, T]$, has the same boundary values as the chosen trajectory of the second family, and is close to it in terms of control. Make a change of variable on the interval $[-T, 0]$ leaving the point $-T$ unchanged and making the $u_0$-component of the control equal to 1. Then the interval $[-T, 0]$ transforms into $[-T, \varepsilon]$, the $x_0$-component of the trajectory becomes equal to $t$, and the function $x_2(t)$ satisfies the equation $\dot{x}_2 = u_1$, with the function $u_1(t)$ being small on $[-T, \varepsilon]$. The function $x_1(t)$ will satisfy the equation

$$\dot{x}_1 = u_1 \cdot \frac{1}{2} t^2 x_2. \tag{10.17}$$

Obviously, $x_2(-T) = 0, x_1(-T) = \tilde{x}_1$.

On the interval $[0, T]$ we make a change of variable such that the point $T$ stays unchanged and $u_1(t) = 1$. Then the interval $[0, T]$ transforms into $[\eta, T]$, the $x_2$-component of the trajectory transforms into $t$, and the $x_0$-component will satisfy the equation $\dot{x}_0 = u_0(t)$, with $u_0(t)$ being small on $[\eta, T]$, and the condition $x_0(T) = 0$. The $x_1$-component will satisfy the equation $\dot{x}_1 = \frac{1}{2} t x_0^2$ with boundary condition $x_1(T) = \tilde{x}_1$. From continuity conditions we obtain

$$x_2(\varepsilon) = \eta, \quad x_0(\eta) = \varepsilon, \quad x_1(\varepsilon) = x_1(\eta).$$

We will show that such a trajectory is only possible for

$$\varepsilon = \eta = 0, \quad x_2(t) = 0 \mid [-T, 0], \quad x_0(t) = 0 \mid [0, T].$$

This will prove the rigidity of a trajectory of the second family.

Integrating by parts we obtain

$$x_1(\varepsilon) - \tilde{x}_1 = \frac{1}{4} \varepsilon^2 \eta^2 - \frac{1}{2} \int_{-T}^\varepsilon t x_2^2(t) \, dt.$$
Let $\varepsilon \neq 0$. We will show that in this case $|\eta| > |\varepsilon|$. Indeed, assume that $|\eta| \leq |\varepsilon|$. For definiteness, let $\varepsilon > 0, \eta < 0$. The other cases are treated similarly. We have

$$-\frac{1}{2} \int_{-T}^{\varepsilon} x_2(t) \, dt = -\frac{1}{2} \int_{-T}^{0} x_2(t) \, dt - \frac{1}{2} \int_{0}^{\varepsilon} x_2(t) \, dt \geq -\frac{1}{2} \int_{0}^{\varepsilon} x_2(t) \, dt.$$ 

But on the interval $[0, \varepsilon]$

$$|x_2(t)| \leq |\eta| + \delta \varepsilon \leq \varepsilon (1 + \delta).$$

Hence

$$(10.18) \quad -\frac{1}{2} \int_{-T}^{\varepsilon} x_2(t) \, dt \geq -\frac{1}{4} (1 + \delta)^2 \varepsilon^4.$$ 

Further, we have

$$\frac{1}{2} \int_{\eta}^{T} x_2(t) \, dt = \frac{1}{2} \int_{\eta}^{0} x_2(t) \, dt + \frac{1}{2} \int_{0}^{T} x_2(t) \, dt.$$ 

On the interval $[\eta, 0]$

$$|x_0(t)| \leq \varepsilon + \delta |\eta| \leq (1 + \delta) \varepsilon.$$ 

Hence

$$(10.19) \quad \frac{1}{2} \int_{\eta}^{0} x_2(t) \, dt \geq -\frac{1}{4} (1 + \delta)^2 \varepsilon^2 \eta^2.$$ 

Further, $x_0(0) \geq \varepsilon - \delta |\eta| \geq (1 - \delta) \varepsilon$. Therefore

$$(10.20) \quad \frac{1}{2} \int_{0}^{T} x_2(t) \, dt \geq \frac{1}{2} \int_{0}^{1} \left( (1 - \delta) \varepsilon - \delta t \right) dt = \frac{1}{2} \left( \frac{1 - \delta}{\delta^2} \right) \varepsilon^4 \int_{0}^{1} \tau (1 - \tau) \, d\tau.$$ 

We see from (10.18), (10.19), and (10.20) that the left-hand side of (10.17) is no less than

$$-\frac{1}{4} \varepsilon^2 \eta^2 - \frac{1}{4} (1 + \delta)^2 \varepsilon^4 - \frac{1}{4} (1 + \delta)^2 \varepsilon^2 \eta^2 + \frac{1}{2} \left( \frac{1 - \delta}{\delta^2} \right) \varepsilon^4 \int_{0}^{1} \tau (1 - \tau) \, d\tau.$$ 

But the last expression is positive for small $\delta$, since $|\eta| \leq \varepsilon$. Thus we have proved that $\varepsilon \neq 0 \implies |\eta| > |\varepsilon|$. One can establish in a quite similar way that if $\eta \neq 0$ then $|\varepsilon| > |\eta|$. Thus for small $\delta$ the equality (10.17) can hold only for $\varepsilon = \eta = 0$. Hence it obviously follows that $x_2(t) = 0 \mid [-T, 0]$ and $x_0(t) = 0 \mid [0, T]$. Thus the rigidity of a trajectory of the second family is proved. 

Actually we have found the description of all rigid trajectories in the case (10.16). These are broken extremals and smooth extremals that either do not pass through the singular point or begin and end at this point.

This example is of interest in that it contains certain positive information about non-smooth, broken extremals, and about extremals which do not belong to the main stratum. These were the concluding remarks which appeared relevant to us.

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